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Bernard Coupet, Hervé Gaussier, Alexandre Sukhov

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BERNARD COUPET, HERVÉ GAUSSIER AND ALEXANDRE SUKHOV

Abstract. We give a necessary and sufficient condition for the smooth extension of a diffeomorphism between smooth strictly pseudoconvex domains in four real dimensional almost complex manifolds (see Theorem 1.1). The proof is mainly based on a reflection principle for pseudoholomorphic discs, on precise estimates of the Kobayashi-Royden infinitesimal pseudometric and on the scaling method in almost complex manifolds.

1. Introduction

The analysis on almost complex manifolds, first developed by Newlander-Nirenberg and Nijenhuis-Woolf, appeared crucial in symplectic and contact geometry with the fundamental work of M.Gromov [17]. Since the literature dedicated to this subject is rapidly growing we just mention the book [1] and references therein. The (geometric) analysis on almost complex manifolds became one of the most powerful tools in symplectic geometry, making its systematic developpement relevant. The present paper is a step in this program.

Fefferman's mapping theorem [13] states that a biholomorphism between two smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^n$ extends as a smooth diffeomorphism between their closures. This result had a strong impact on the developpement of complex analysis on domains in $\mathbb{C}^n$ during the last twenty five years. Our main goal is to prove an analogue of this theorem in almost complex manifolds. Complex and symplectic structures are usually related as follows. Let (M,ω) and (M',ω') be two real manifolds equipped with symplectic forms and let J be an almost complex structure on M tamed by ω (so that $\omega(v,Jv) > 0$ for any non-zero vector v). If $\phi : M \to M'$ is a symplectomorphism, the direct image $J' := \phi_*(J) = d\phi \circ J \circ d\phi^{-1}$ of the structure J is an almost complex structure on M' tamed by ω' and ϕ is a biholomorphism with respect to J and J'. This property enables to construct topological invariants of symplectic structures employing the complex geometry. In his survey [2], D.Bennequin raised the question of a symplectic analogue of this theorem. E.Chirka constructed in [7] an example of a symplectomorphism of the unit ball in $\mathbb{C}^n$ with the usual symplectic structure, having a wild boundary behaviour. This gives a negative answer to the question. Our main result shows that Fefferman's theorem remains true in the category of almost complex manifolds :

Theorem 1.1. *Let D and D' be two smooth relatively compact domains in real four dimensional manifolds. Assume that D admits an almost complex structure J smooth on $\bar{D}$ and such that (D,J) is strictly pseudoconvex. Then a smooth diffeomorphism $f : D \to D'$ extends to a smooth diffeomorphism between $\bar{D}$ and $\bar{D}'$ if and only if the direct image $f_*(J)$ of J under f extends smoothly on $\bar{D}'$ and $(D', f_*(J))$ is strictly pseudoconvex.*

One can see the smooth extension of the direct image $f_*(J)$ to the boundary of D' as the smooth extension, up to the boundary, of a part of first order partial derivatives of the components of f.

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Theorem 1.1 claims that all the partial derivatives necessarily extend smoothly up to the boundary. This statement is a geometric version of the elliptic regularity and is a criterion applicable (at least in principle) to any diffeomorphism between four dimensional real manifolds with boundaries.

Theorem 1.1 admits the following formulation, closer to the claasical one.

Theorem 1.2. *A biholomorphism between two smooth relatively compact strictly pseudoconvex domains in (real) four dimensional almost complex manifolds extends to a smooth diffeomorphism between their closures.*

Theorem 1.2 is a generalization of Fefferman's mapping theorem in (complex) dimension 2. We point out that in the almost complex category, the real four dimensional manifolds often represent the most interesting class from the point of view of symplectic geometry. Our restriction on the dimension comes from our method of proof; we do not know if it necessary in the general case.

The original proof of C.Fefferman is based on a subtil investigation of asymptotic behavior of the Bergman kernel in strictly pseudoconvex domains. Later several different approaches have been proposed. Similarly to the integrable case (see, for instance, L.Nirenberg-S.Webster-P.Yang [23], S.Pinchuk-S.Khasanov [26], B.Coupet [9] and F.Forstneric [14]) our proof is based on boundary estimates of the infinitesimal Kobayashi- Royden pseudometric and on the smooth reflection principle. We point out that in the almost complex case, the reflection principle for totally real manifolds has been used by H.Hofer [19], S.Ivashkovich-V.Shevchishin [20], E.Chirka [8].

The paper is organized as follows.

Section 2 is preliminary and contains general facts about almost complex manifolds. In Subsection 2.5 for a given point on a strictly pseudoconvex hypersurface Γ in a four dimensional almost complex manifold (M, J), we construct a coordinates chart in which J is a sufficiently small diagonal perturbation of the standard structure J_{st} on $\mathbb{C}^2$ and Γ is strictly pseudoconvex with respect to J_{st}. Such a representation will be crucially used and explains the restriction to the four dimensional case in our approach.

In Section 3 we prove that in the hypothesis of Theorem 1.1 a biholomorphism f extends as a 1/2-Hölder map between the closures of the domains and we study the boundary behaviour of its tangent map. Similarly to the integrable case, our proof is based on the Hopf lemma and the estimates of the Kobayashi-Royden infinitesimal pseudometric obtained in [15]. This result allows to restrict our considerations to the case where f is a biholomorphism between two strictly pseudoconvex domains in $\mathbb{C}^2$ with small almost complex deformations of the standard structure.

Sections 4 and 5 contain another technical ingredient necessary for the proof of Fefferman's theorem : results on the boundary regularity of pseudoholomorphic maps near totally real manifolds. In Section 4 we study the boundary regularity of a pseudoholomorphic disc attached, in the sense of the cluster set, to a totally real submanifold of an almost complex manifold. The proof consists of two steps. First we obtain an a priori bound for the gradient of the disc using uniform estimates of the Kobayashi-Royden infinitesimal pseudometric in the Grauert tube around a totally real manifold (in the integrable case a similar construction has been used in [6]). Then we apply a version of the smooth reflection principle in almost complex manifolds (this construction is due to E.Chirka [8]). In Section 5 we establish the boundary regularity of a pseudoholomorphic map defined in a wedge with a totally real edge and taking this edge (in the sense of the cluster set) to a totally real submanifold in an almost complex manifold. For the proof we fill the wedge by pseudoholomorphic discs attached to the edge along the upper semi circle (this generalizes Pinchuk's construction in the integrable case [24]) and we apply the results of Section 4.

In Section 6 we show how to deduce the proof of Fefferman's theorem from the results of Section 5. The main idea is to consider the cotangent lift $\tilde{f}$ of a biholomorphism f. According to the known results of the differential geometry [32], an almost complex structure J on M admits a canonical almost complex lift $\tilde{J}$ to the cotangent bundle of M such that $\tilde{f}$ is biholomorphic with respect to $\tilde{J}$ and the conormal bundle of a strictly J-pseudoconvex hypersurface is totally real with respect to $\tilde{J}$. In the integrable case the holomorphic tangent bundle of a strictly pseudoconvex hypersurface (that is the projectivization of the conormal bundle) is frequently used instead of the conormal bundle (see [31, 26, 9]). Since in the almost complex case the projectivization of the cotangent bundle does not admit a natural almost complex structure, we need to deal with the conormal bundle, similarly to the ideas of A.Tumanov [29]. In the case where f is of class $\mathcal{C}^1$ up to the boundary of D, its cotangent lift extends continuously to the conormal bundle of ∂D and takes it to the conormal bundle of $\partial D'$, which implies Fefferman's theorem in that case in view of the results of Section 5.

In Section 7 we consider the general situation of Theorem 1.1. We prove that the cotangent lift of f takes the conormal bundle of ∂D to the conormal bundle of $\partial D'$ in the sense of the cluster set. This is sufficient in order to apply the results of Section 5. Our proof is based on the results of Section 3 and the scaling method introduced by S.Pinchuk [25] in the integrable case which we develop in our situation.

2. Preliminaries

2.1. **Almost complex manifolds.** Let (M', J') and (M, J) be almost complex manifolds and let f be a smooth map from M' to M. We say that f is (J', J)-*holomorphic* if $df \circ J' = J \circ df$ on TM'. We denote by $\mathcal{O}_{(J',J)}(M', M)$ the set of (J', J)-holomorphic maps from M' to M. Let Δ be the unit disc in $\mathbb{C}$ and J_{st} be the standard integrable structure on $\mathbb{C}^n$ for every n. If $(M', J') = (\Delta, J_{st})$, we denote by $\mathcal{O}_J(\Delta, M)$ the set $\mathcal{O}_{(J_{st},J)}(\Delta, M)$ of J-*holomorphic discs* in M.

The following Lemma shows that every almost complex manifold (M, J) can be viewed locally as the unit ball in $\mathbb{C}^n$ equipped with a small almost complex deformation of J_{st}. This will be used frequently in the sequel.

Lemma 2.1. *Let (M, J) be an almost complex manifold. Then for every point $p \in M$ and every $\lambda_0 > 0$ there exist a neighborhood U of p and a coordinate diffeomorphism $z : U \to \mathbb{B}$ such that $z(p) = 0$, $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$ and the direct image $\hat{J} = z_*(J)$ satisfies $||\hat{J} - J_{st}||_{\mathcal{C}^2(\bar{\mathbb{B}})} \leq \lambda_0$.*

Proof. There exists a diffeomorphism z from a neighborhood U' of $p \in M$ onto $\mathbb{B}$ satisfying $z(p) = 0$ and $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$. For $\lambda > 0$ consider the dilation $d_\lambda : t \mapsto \lambda^{-1} t$ in $\mathbb{C}^n$ and the composition $z_\lambda = d_\lambda \circ z$. Then $\lim_{\lambda \to 0} ||(z_\lambda)_*(J) - J_{st}||_{\mathcal{C}^2(\bar{\mathbb{B}})} = 0$. Setting $U = z_\lambda^{-1}(\mathbb{B})$ for $\lambda > 0$ small enough, we obtain the desired statement. □

2.2. ∂_J **and** $\bar{\partial}_J$ **operators.** Let (M, J) be an almost complex manifold. We denote by TM the real tangent bundle of M and by $T_{\mathbb{C}}M$ its complexification. Recall that $T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$ where $T^{(1,0)}M := \{X \in T_{\mathbb{C}}M : JX = iX\} = \{\zeta - iJ\zeta, \zeta \in TM\}$, and $T^{(0,1)}M := \{X \in T_{\mathbb{C}}M : JX = -iX\} = \{\zeta + iJ\zeta, \zeta \in TM\}$. Let T^*M denote the cotangent bundle of M. Identifying $\mathbb{C} \otimes T^*M$ with $T^*_{\mathbb{C}}M := Hom(T_{\mathbb{C}}M, \mathbb{C})$ we define the set of complex forms of type $(1, 0)$ on M by : $T^*_{(1,0)}M = \{w \in T^*_{\mathbb{C}}M : w(X) = 0, \forall X \in T^{(0,1)}M\}$ and the set of complex forms of type $(0, 1)$ on M by : $T^*_{(0,1)}M = \{w \in T^*_{\mathbb{C}}M : w(X) = 0, \forall X \in T^{(1,0)}M\}$. Then $T^*_{\mathbb{C}}M = T^*_{(1,0)}M \oplus T^*_{(0,1)}M$. This allows to define the operators ∂_J and $\bar{\partial}_J$ on the space of smooth functions defined on M : given a complex smooth function u on M, we set $\partial_J u = du_{(1,0)} \in T^*_{(1,0)}M$ and $\bar{\partial}_J u = du_{(0,1)} \in T^*_{(0,1)}M$. As usual, differential forms of any bidegree (p, q) on (M, J) are defined by means of the exterior product.

2.3. **Real submanifolds in an almost complex manifold.** Let Γ be a real smooth submanifold in M and let $p \in \Gamma$. We denote by $H^J(\Gamma)$ the J-holomorphic tangent bundle $T\Gamma \cap JT\Gamma$.

Definition 2.2. The real submanifold Γ is totally real if $H^J(\Gamma) = \{0\}$.

We note that if Γ is a real hypersurface in M defined by $\Gamma = \{r = 0\}$ and $p \in \Gamma$ then by definition $H^J_p(\Gamma) = \{v \in T_pM : dr(p)(v) = dr(p)(J(p)v) = 0\}$.

We recall the notion of the Levi form of a hypersurface :

Definition 2.3. Let $\Gamma = \{r = 0\}$ be a smooth real hypersurface in M (r is any smooth defining function of Γ) and let $p \in \Gamma$.

(i) The *Levi form* of Γ at p is the map defined on $H^J_p(\Gamma)$ by $\mathcal{L}^J_\Gamma(X_p) = J^\star dr[X, JX]_p$, where the vector field X is any section of the J-holomorphic tangent bundle $H^J\Gamma$ such that $X(p) = X_p$.

(ii) A real smooth hypersurface $\Gamma = \{r = 0\}$ in M is *strictly J-pseudoconvex* if its Levi form $\mathcal{L}^J_\Gamma$ is positive definite on $H^J(\Gamma)$.

Remark 2.4. (i) the "strict J-pseudoconvexity" condition does not depend on the choice of a smooth defining function of Γ. Indeed if ρ is an other smooth defining function for Γ in a neighborhood of $p \in \Gamma$ then there exists a positive smooth function λ defined in a neighborhood of p such that $\rho = \lambda r$. In particular $(J^\star dr)(p) = \lambda(p)(J^\star d\rho)(p)$.

(ii) since the map $(r, J) \mapsto J^\star dr$ is smooth the "strict J-pseudoconvexity" is stable under small perturbations of both the hypersurface and the almost complex structure.

Let $X \in TM$. It follows from the identity $d(J^\star dr)(X, JX) = X(< J^\star dr, JX >) - JX(< J^\star dr, X >) - (J^\star dr)[X, JX]$ that $(J^\star dr)[X, JX] = -d(J^\star dr)(X, JX)$ for every $X \in H^J\Gamma$, since $< dr, JX >=< dr, JX >= 0$ in that case. Hence we set

Definition 2.5. If r is a $\mathcal{C}^2$ function on M then the Levi form of r is defined on TM by $\mathcal{L}^J(r)(X) := -d(J^\star dr)(X, JX)$.

Let $p \in M$ and $v \in T_pM$. We will denote by $\mathcal{L}^J(r)(p)(v)$ the quantity $\mathcal{L}^J(r)(X)(p)$ where X is any section of TM such that $X(p) = v$. Obviously, the Levi form $\mathcal{L}^J(r)$ is determined by the form $-d(J^\star dr)_{(1,1)}$ (where the $(1,1)$ part of $-d(J^\star dr)(X, JX)$ is taken with respect to J).

2.4. **Kobayashi-Royden infinitesimal pseudometric.** Let (M, J) be an almost complex manifold. In what follows we use the notation $\zeta = x + iy \in \mathbb{C}$. According to [22], for every $p \in M$ there is a neighborhood $\mathcal{V}$ of 0 in T_pM such that for every $v \in \mathcal{V}$ there exists $f \in \mathcal{O}_J(\Delta, M)$ satisfying $f(0) = p$, $df(0)(\partial/\partial x) = v$. This allows to define the Kobayashi-Royden infinitesimal pseudometric $K_{(M,J)}$.

Definition 2.6. For $p \in M$ and $v \in T_pM$, $K_{(M,J)}(p, v)$ is the infimum of the set of positive α such that there exists a J-holomorphic disc $f : \Delta \to M$ satisfying $f(0) = p$ and $df(0)(\partial/\partial x) = v/\alpha$.

Since for every $f \in \mathcal{O}_{(J',J)}(M', M)$ and every $\varphi \in \mathcal{O}_J(\Delta, M')$ the composition $f \circ \varphi$ is in $\mathcal{O}_J(\Delta, M)$ we have :

Proposition 2.7. *Let $f : (M', J') \to (M, J)$ be a (J', J)-holomorphic map. Then $K_{(M,J)}(f(p'), df(p')(v')) \leq K_{(M',J')}(p', v')$ for every $p' \in M'$, $v' \in T_{p'}M'$.*

2.5. **Plurisubharmonic functions.** We first recall the following definition :

Definition 2.8. An upper semicontinuous function u on (M, J) is called *J-plurisubharmonic* on M if the composition $u \circ f$ is subharmonic on Δ for every $f \in \mathcal{O}_J(\Delta, M)$.

If M is a domain in $\mathbb{C}^n$ and $J = J_{st}$ then a J_{st}-plurisubharmonic function is a plurisubharmonic function in the usual sense.

The next proposition gives a characterization of J-plurisubharmonic functions (see [11, 18]) :

Proposition 2.9. *Let u be a $\mathcal{C}^2$ real valued function on M. Then u is J-plurisubharmonic on M if and only if $\mathcal{L}^J(u)(X) \geq 0$ for every $X \in TM$.*

Proposition 2.9 leads to the definition :

Definition 2.10. A $\mathcal{C}^2$ real valued function u on M is *strictly J-plurisubharmonic* on M if $\mathcal{L}^J(u)$ is positive definite on TM.

We have the following example of a J-plurisubharmonic function on an almost complex manifold (M, J) :

Example 2.11. *For every point $p \in (M, J)$ there exists a neighborhood U of p and a diffeomorphism $z : U \to \mathbb{B}$ centered at p (ie $z(p) = 0$) such that the function $|z|^2$ is J-plurisubharmonic on U.*

Proof. Let $p \in M$, U_0 be a neighborhood of p and $z : U_0 \to \mathbb{B}$ be local complex coordinates centered at p, such that $dz \circ J(p) \circ dz^{-1} = J_{st}$ on $\mathbb{B}$. Consider the function $u(q) = |z(q)|^2$ on U_0. For every $w, v \in \mathbb{C}^n$ we have $\mathcal{L}^{J_{st}}(u \circ z^{-1})(w)(v) = \|v\|^2$. Let $B(0, 1/2)$ be the ball centered at the origin with radius $1/2$ and let $\mathcal{E}$ be the space of smooth almost complex structures defined in a neighborhood of $\overline{B(0, 1/2)}$. Since the function $(J', w) \mapsto \mathcal{L}^{J'}(u \circ z^{-1})(w)$ is continuous on $\mathcal{E} \times B(0, 1/2)$, there exist a neighborhood V of the origin and positive constants λ_0 and c such that $\mathcal{L}^{J'}(u \circ z^{-1})(q)(v) \geq c\|v\|^2$ for every $q \in V$ and for every almost complex structure J' satisfying $\|J' - J_{st}\|_{\mathcal{C}^2(\bar{V})} \leq \lambda_0$. Let U_1 be a neighborhood of p such that $\|z_*(J) - J_{st}\|_{\mathcal{C}^2(\overline{z(U_1)})} \leq \lambda_0$ and let $0 < r < 1$ be such that $B(0, r) \subset V$ and $U := z^{-1}(B(0, r)) \subset U_1$. Then we have the following estimate for every $q \in U$ and $v \in T_qM$: $\mathcal{L}^J(u)(q)(v) \geq c\|v\|^2$. Then $r^{-1}z$ is the desired diffeomorphism. $\square$

We also have the following

Lemma 2.12. *A function u of class $\mathcal{C}^2$ in a neighborhood of a point p of (M, J) is strictly J-plurisubharmonic if and only if there exists a neighborhood U of p with local complex coordinates $z : U \to \mathbb{B}$ centered at p, such that the function $u - c|z|^2$ is J-plurisubharmonic on U for some constant $c > 0$.*

2.6. **Local description of strictly pseudoconvex domains.** If Γ is a germ of a real hypersurface in $\mathbb{C}^n$ strictly pseudoconvex with respect to J_{st}, then Γ remains strictly pseudoconvex for any almost complex structure J sufficiently close to J_{st} in the $\mathcal{C}^2$-norm. Conversely a strictly pseudoconvex hypersurface in an almost complex manifold of real dimension four can be represented, in suitable local coordinates, as a strictly J_{st}-pseudoconvex hypersurface equipped with a small deformation of the standard structure. Indeed, according to [28] Corollary 3.1.2, there exist a neighborhood U of q in M and complex coordinates $z = (z^1, z^2) : U \to B \subset \mathbb{C}^2$, $z(q) = 0$ such that $z_*(J)(0) = J_{st}$ and moreover, a map $f : \Delta \to B$ is $J' := z_*(J)$-holomorphic if it satisfies the equations

$$\frac{\partial f^j}{\partial \bar{\zeta}} = A_j(f^1, f^2)\overline{\left(\frac{\partial f^j}{\partial \zeta}\right)}, j = 1, 2 \tag{2.1}$$

where $A_j(z) = O(|z|)$, $j = 1, 2$.

In order to obtain such coordinates, one can consider two transversal foliations of the ball $\mathbb{B}$ by J'-holomorphic curves (see [22])and then take these curves into the lines $z^j = const$ by a local diffeomorphism. The direct image of the almost complex structure J under such a diffeomorphism has a diagonal matrix $J'(z^1, z^2) = (a_{jk}(z))_{jk}$ with $a_{12} = a_{21} = 0$ and $a_{jj} = i + \alpha_{jj}$ where $\alpha_{jj}(z) = \mathcal{O}(|z|)$ for $j = 1, 2$. We point out that the lines $z^j = const$ are J-holomorphic after a suitable parametrization (which, in general, is not linear).

In what follows we omit the prime and denote this structure again by J. We may assume that the complex tangent space $T_0(\partial D) \cap J(0)T_0(\partial D) = T_0(\partial D) \cap iT_0(\partial D)$ is given by $\{z^2 = 0\}$. In particular, we have the following expansion for the defining function ρ of D on U : $\rho(z, \bar{z}) = 2Re(z^2) + 2ReK(z) + H(z) + \mathcal{O}(|z|^3)$, where $K(z) = \sum k_{\nu\mu} z^\nu z^\mu$, $k_{\nu\mu} = k_{\mu\nu}$ and $H(z) = \sum h_{\nu\mu} z^\nu \bar{z}^\mu$, $h_{\nu\mu} = \bar{h}_{\mu\nu}$.

Lemma 2.13. *The domain D is strictly J_{st}-pseudoconvex near the origin.*

Proof of Lemma 2.13. Consider a complex vector $v = (v_1, 0)$ tangent to ∂D at the origin. Let $f : \Delta \to \mathbb{C}^2$ be a J-holomorphic disc centered at the origin and tangent to v: $f(\zeta) = v\zeta + \mathcal{O}(|\zeta|^2)$. Since $A_2 = \mathcal{O}(|z|)$, it follows from the J-holomorphy equation (2.1) that $(f^2)_{\zeta\bar{\zeta}}(0) = 0$. This implies that $(\rho \circ f)_{\zeta\bar{\zeta}}(0) = H(v)$. Thus, the Levi form with respect to J coincides with the Levi form with respect to J_{st} on the complex tangent space of ∂D at the origin. This proves Lemma 2.13. □

Consider the non-isotropic dilations $\Lambda_\delta : (z^1, z^2) \mapsto (\delta^{-1/2} z^1, \delta^{-1} z^2) = (w^1, w^2)$ with $\delta > 0$. If J has the above diagonal form in the coordinates (z^1, z^2) in $\mathbb{C}^2$, then its direct image $J_\delta = (\Lambda_\delta)_*(J)$ has the form $J_\delta(w^1, w^2) = (a_{jk}(\delta^{1/2} w^1, \delta w^2))_{jk}$ and so J_δ tends to J_{st} in the $\mathcal{C}^2$ norm as $\delta \to 0$. On the other hand, ∂D is, in the w coordinates, the zero set of the function $\rho_\delta = \delta^{-1}(\rho \circ \Lambda_\delta^{-1})$. As $\delta \to 0$, the function ρ_δ tends to the function $2Rew^2 + 2ReK(w^1, 0) + H(w^1, 0)$ which defines a strictly J_{st}-pseudoconvex domain by Lemma 2.13 and proves the claim.

This also proves that if ρ is a local defining function of a strictly J-pseudoconvex domain, then $\tilde{\rho} := \rho + C\rho^2$ is a strictly J-plurisubharmonic function, quite similarly to the standard case.

In conclusion we point out that extending $\tilde{\rho}$ by a suitable negative constant, we obtain that if D is a strictly J-pseudoconvex domain in an almost complex manifold, then there exists a neighborhood U of $\bar{D}$ and a function ρ, J-plurisubharmonic on U and strictly J-plurisubharmonic in a neighborhood of ∂D, such that $D = \{\rho < 0\}$.

3. Boundary continuity and localization of biholomorphisms

In this section we give some preliminary technical results necessary for the proof of Theorem 1.1.

3.1. **Hopf lemma and the boundary distance preserving property.**

In what follows we need an analog of the Hopf lemma for almost complex manifolds. It can be proved quite similarly to the standard one.

Lemma 3.1. *(Hopf lemma) Let G be a relatively compact domain with a $\mathcal{C}^2$ boundary on an almost complex manifold (M, J). Then for any negative J-psh function u on D there exists a constant $C > 0$ such that $|u(p)| \geq Cdist(p, \partial G)$ for any $p \in G$ (dist is taken with respect to a Riemannian metric on M).*

Proof of Lemma 3.1. Step 1. We have the following precise version on the unit disc: let u be a subharmonic function on Δ, K be a fixed compact on Δ. Suppose that $u < 0$ on Δ and $u|K \leq -L$ where $L > 0$ is constant. Then there exists $C(K, L) > 0$ (independent of u) such that $|u(p)| \geq Cdist(p, \partial\Delta)$ (see [27]).

Step 2. Let G be a domain in $\mathbb{C}$ with $\mathcal{C}^2$-boundary. Then there exists an $r > 0$ (depending on the curvature of the boundary) such that for any boundary point $q \in \partial G$ the ball $B_{q,r}$ of radius r centered on the interior normal to ∂G at q, such that $q \in \partial B_{q,r}$, is contained in G. Applying Step 1 to the restriction of u on every such a ball (when q runs over ∂G) we obtain the Hopf lemma for a domain with $\mathcal{C}^2$ boundary: let u be a subharmonic function on G, K be a fixed compact on G. Suppose that $u < 0$ on G and $u|K \leq -L$ where $L > 0$ is constant. Denote by k the curvature of ∂G. Then there exists $C(K, L, k) > 0$ (independent of u) such that $|u(p)| \geq C dist(p, \partial\Delta)$.

Step 3. Now we can prove the Hopf lemma for almost complex manifolds. Fix a normal field v on ∂G and consider the family of J-holomorphic discs d_v satisfying $d'_0(\partial_x) = v(d(0))$. The image of such a disc is a real surfaces intesecting ∂G transversally, so its pullback gives a $\mathcal{C}^2$-curve in Δ. Denote by G_v the component of Δ defined by the condition $d_v(G_v) \subset G$. Then every G_v is a domain with $\mathcal{C}^2$-boundary in $\mathbb{C}$ and the curvatures of boundaries depend continuously on v. We conclude by applying Step 2 to the composition $u \circ d_v$ on G_v.

As an application, we obtain the boundary distance preserving property for biholomorphisms between strictly pseudoconvex domains.

Proposition 3.2. *Let D and D' be two smoothly bounded strictly pseudoconvex domains in four dimensional almost complex manifolds (M, J) and (M', J') respectively and let $f : D \rightarrow D'$ be a (J, J')-biholomorphism. Then there exists a constant $C > 0$ such that*

$$(1/C) dist(f(z), \partial D') \leq dist(z, \partial D) \leq C dist(f(z), \partial D').$$

Proof of Proposition 3.2. According to the previous section, we may assume that $D = \{p : \rho(p) < 0\}$ where ρ is a J-plurisubharmonic function on D, strictly J-plurisubharmonic in a neighborhood of the boundary; similarly D' can be defined by means of a function ρ'. Now it suffices to apply the Hopf lemma to the functions $\rho' \circ f$ and $\rho \circ f^{-1}$. □

3.2. **Boundary continuity of diffeomorphisms.** Using estimates of the Kobayashi-Royden metric together with the boundary distance preserving property, we obtain, by means of classical arguments (see, for instance, K.Diederich-J.E.Fornaess [12]), the following

Proposition 3.3. *Let D and D' be two smoothly relatively compact strictly pseudoconvex domains in almost complex manifolds (M, J) and (M', J') respectively. Let $f : D \rightarrow D'$ be a smooth diffeomorphism biholomorphic with respect to J and J'. Then f extends as a $1/2$-Hölder homeomorphism between the closures of D and D'.*

As usual, we denote by $K_{(D,J)}(p, v)$ the value of the Kobayashi-Royden infinitesimal metric (with respect to the structure J) at a point p and a tangent vector v. We begin with the folllowing estimates of the Kobayashi-Royden infinitesimal metric :

Lemma 3.4. *Let D be a relatively compact strictly pseudoconvex domain in an almost complex manifold (M, J). Then there exists a constant $C > 0$ such that*

$$(1/C)\|v\|/dist(p, \partial D)^{1/2} \leq K_{(D,J)}(p, v) \leq C\|v\|/dist(p, \partial D)$$

for every $p \in D$ and $v \in T_pM$.

Proof of Lemma 3.4. The lower estimate is proved in [15]. For the upper estimate, it is sufficient to prove the statement near the boundary. Let $q \in \partial D$. One may suppose that $q = 0$ and $J = J_{st} + 0(|z|)$. For $p \in U \cap D$ consider the ball $p + d(p)\mathbb{B}$, where $d(p)$ is the distance from p to ∂D. It follows by A.Nijenhuis-W.Woolf [22] that there exists constant $C_1, C_2 > 0$ and a function $d'(p)$ satisfying $C_1 d'(p) \leq d(p) \leq C_2 d'(p)$ on $D \cap U$, such that for any complex vector v there exists a J holomorphic map $f : d'(p)\Delta \rightarrow p + d(p)\mathbb{B}$ such that $f(0) = p$ and $df_0(e) = v/\|v\|$ (e is the unit vector 1 in $\mathbb{C}$). By the decreasing property of the Kobayashi-Royden metric we obtain that $K_{(D,J)}(z, v/\|v\|) \leq K_{d'(p)\Delta}(0, e)$ which implies the desired estimate.

Proof of Proposition 3.3. For any $p \in D$ and any tangent vector v at p we have by Lemma 3.4 :

$$C_1 \frac{\|df_p(v)\|}{dist(f(p), \partial D')^{1/2}} \leq K_{(D',J')}(f(p), df_p(v)) = K_{(D,J)}(p,v) \leq C_2 \frac{\|v\|}{dist(p, \partial D)}$$

which implies, by Proposition 3.2, the estimate

$$|||df_p||| \leq C \frac{\|v\|}{dist(p, \partial D)^{1/2}}.$$

This gives the desired statement. □

Proposition 3.3 allows to reduce the proof of Fefferman's theorem to a *local situation.* Indeed, let p be a boundary point of D and $f(p) = p' \in \partial D'$. It suffices to prove that f extends smoothly to a neighborhood of p on ∂D. Consider coordinates z and z' defined in small neighborhoods U of p and U' of p' respectively, with $U' \cap D' = f(D \cap U)$ (this is possible since f extends as a homeomorphism at p). We obtain the following situation. If $\Gamma = z(\partial D \cap U)$ and $\Gamma' = z'(\partial D' \cap U')$ then the map $z' \circ f \circ z^{-1}$ is defined on $z(D \cap U)$ in $\mathbb{C}^2$, continuous up to the hypersurface Γ with $f(\Gamma) \subset \Gamma'$. Furthermore the map $z' \circ f \circ z^{-1}$ is a diffeomorphism between $z(D \cap U)$ and $z'(D' \cap U')$ and the hypersurfaces Γ and Γ' are strictly pseudoconvex for the structures $z_*(J)$ and $(z')_*(J')$ respectively. Finally, we may choose z and z' such that $z_*(J)$ and $z'_*(J')$ are represented by diagonal matrix functions in the coordinates z and z'. As we proved in Lemma 2.13, Γ (resp. Γ') is also strictly J_{st}-psdeudoconvex at the origin. We call such coordinates z (resp. z') *canonical coordinates* at p (resp. at p'). Using the non-isotropic dilation as in Section 2.5, we may assume that the norms $\|z_*(J) - J_{st}\|_{\mathcal{C}^2}$ and $\|z'_*(J') - J_{st}\|_{\mathcal{C}^2}$ are as small as needed. This localization is crucially used in the sequel and we write J (resp. J') instead of $z_*(J)$ (resp. $z'_*(J')$); we identify f with $z' \circ f \circ z^{-1}$.

3.3. **Localization and boundary behavior of the tangent map.** In what follows we will need a more precise information about the boundary behavior of the tangent map of f. Recall that according to the previous subsection, D and D' are supposed to be domains in $\mathbb{C}^2$, Γ and Γ' are open smooth pieces of their boundaries containing the origin, the almost complex structure J (resp. J') is defined in a neighborhood of D (resp. D'), f is a (J, J') biholomorphism from D to D', continuous up to Γ, $f(\Gamma) = \Gamma'$, $f(0) = 0$. The matrix J (resp. J') is diagonal on D (resp. D').

Consider a basis (ω_1, ω_2) of $(1,0)$ differential forms (for the structure J) in a neighborhood of the origin. Since J is diagonal, we may choose $\omega_j = dz^j - B_j(z) d\bar{z}^j$, $j = 1, 2$. Denote by $Y = (Y_1, Y_2)$ the corresponding dual basis of $(1,0)$ vector fields. Then $Y_j = \partial/\partial z^j - \beta_j(z) \partial/\bar{\partial} z^j$, $j = 1, 2$. Here $\beta_j(0) = \beta_k(0) = 0$. The basis $Y(0)$ simply coincides with the canonical (1,0) basis of $\mathbb{C}^2$. In particular $Y_1(0)$ is a basis vector of the holomorphic tangent space $H^J_0(\partial D)$ and $Y_2(0)$ is "normal" to ∂D. Consider now for $t \geq 0$ the translation $\partial D - t$ of the boundary of D near the origin. Consider, in a neighborhood of the origin, a $(1,0)$ vector field X_1 (for J) such that $X_1(0) = Y_1(0)$ and $X_1(z)$ generates the complex tangent space $H^J_z(\partial D - t)$ at every point $z \in \partial D - t$, $0 \leq t << 1$. Setting $X_2 = Y_2$, we obtain a basis of vector fields $X = (X_1, X_2)$ on D (restricting D if necessary). Any complex tangent vector $v \in T^{(1,0)}_z(D, J)$ at point $z \in D$ admits the unique decomposition $v = v_t + v_n$ where $v_t = \alpha_1 X_1(z)$ (the tangent component) and $v_n = \alpha_2 X_2(z)$ (the normal component). Identifying $T^{(1,0)}_z(D, J)$ with $T_z D$ we may consider the decomposition $v = v_t + v_n$ for $v \in T_z(D)$. Finally we consider this decomposition for points z in a neighborhood of the boundary.

We fix a (1,0) basis vector fields X (resp. X') on D (resp. D') as above.

Proposition 3.5. *The matrix $A = (A_{kj})_{k,j=1,2}$ of the differential df_z with respect to the bases $X(z)$ and $X'(f(z))$ satisfies the following estimates : $A_{11} = O(1)$, $A_{12} = O(dist(z,\partial D)^{-1/2})$, $A_{21} = O(dist(z,\partial D)^{1/2})$ and $A_{22} = O(1)$.*

We begin with the following estimates of the Kobayashi-Royden infinitesimal pseudometric :

Lemma 3.6. *There exists a positive constant C such that for any $p \in D$ and $v \in T_pD$:*

$$\frac{1}{C}\left(\frac{|v_t|}{dist(p,\partial D)^{1/2}} + \frac{|v_n|}{dist(p,\partial D)}\right) \leq K_{(D,J)}(p,v) \leq C\left(\frac{|v_t|}{dist(p,\partial D)^{1/2}} + \frac{|v_n|}{dist(p,\partial D)}\right).$$

Proof of Lemma 3.6. The lower estimate is proved in [15]. For $p \in D$ denote by $d(p)$ the distance from p to ∂D. As in the proof of Lemma 3.4, it follows by Nijenhuis-Woolf [22] that there is $r > 0$, independent of p, such that for any tangent vector v at p satisfying $\omega_2(v) = 0$, there exists a J-holomorphic map $f : r(d(p))^{1/2}\Delta \to D$ satisfying $df_0(e) = v$. This implies the upper estimate. □

Proof of Proposition 3.5. Consider the case where $v = v_t$. It follows from Lemma 3.6 that :

$$\begin{aligned}\frac{1}{C}\left(\frac{\|(df_z(v_t))_t\|}{dist(f(z),\partial D')^{1/2}} + \frac{\|(df_z(v_t))_n\|}{dist(f(z),\partial D')}\right) &\leq K_{(D',J')}(f(z), df_z(v_t)) \\ &= K_{(D,J)}(z,v_t) \leq C\frac{\|v_t\|}{dist(z,\partial D)^{1/2}}.\end{aligned}$$

This implies that $\|(df_z(v_t))_t\| \leq C^{5/2}\|v_t\|$ and $\|(df_z(v_t))_n\| \leq C^3 dist(z,\partial D)^{1/2}\|v_t\|$, by the boundary distance preserving property given in Proposition 3.2. We obtain the estimates for the normal component in a similar way. □

4. Boundary regularity of a pseudoholomorphic disc attached to a totally real manifold

This section is devoted to one of the main technical steps of our construction. We prove that a pseudoholomorphic disc attached (in the sense of the cluster set) to a smooth totally real submanifold in an almost complex manifold, extends smoothly up to the boundary. In the case of the integrable structure, various versions of this statement have been obtained by several authors. In the almost complex case, similar assertions have been established by H.Hofer [19], J.-C.Sikorav [28], S.Ivashkovich-V.Shevchishin [20], E.Chirka [8] under stronger assumptions on the initial boundary regularity of the disc (at least the continuity is required). Our proof consists of two steps. First, we show that a disc extends as a 1/2-Hölder continuous map up to the boundary. The proof is based on special estimates of the Kobayashi-Royden metric in "Grauert tube" type domains. The second step is the reflection principle adapted to the almost complex category; here we follow the construction of E.Chirka [8].

4.1. **Hölder extension of holomorphic discs.** We study the boundary continuity of pseudoholomorphic discs attached to smooth totally real submanifolds in almost complex manifolds.

Recall that in the case of the integrable structure every smooth totally real submanifold E (of maximal dimension) is the zero set of a positive strictly plurisubharmonic function of class $\mathcal{C}^2$. This remains true in the almost complex case. Indeed, we can choose coordinates z in a neighborhood U of $p \in E$ such that $z(p) = 0$, $z_*(J) = J_{st} + O(|z|)$ on U and $z(E \cap U) = \{w = (x,y) \in z(U) : r_j(w) = x_j + o(|w|) = 0\}$. The function $\rho = \sum_{j=1}^n r_j^2$ is strictly J_{st}-plurisubharmonic on $z(U)$ and so remains strictly $z_*(J)$-plurisubharmonic, restricting U if necessary. Covering E by such neighborhoods, we conclude by mean of the partition of unity.

Let $f : \Delta \to (M, J)$ be a J-holomorphic disc and let γ be an open arc on the unit circle $\partial\Delta$. As usual we denote by $C(f, \gamma)$ the cluster set of f on γ; this consists of points $p \in M$ such that $p = \lim_{k\to\infty} f(\zeta_k)$ for a sequence $(\zeta_k)_k$ in Δ converging to a point in γ.

Proposition 4.1. *Let G be a relatively compact domain in an almost complex manifold (M, J) and let ρ be a strictly J-plurisubharmonic function of class $\mathcal{C}^2$ on $\bar{G}$. Let $f : \Delta \to G$ be a J-holomorphic disc such that $\rho \circ f \geq 0$ on Δ. Suppose that γ is an open non-empty arc on $\partial\Delta$ such that the cluster set $C(f, \gamma)$ is contained in the zero set of ρ. Then f extends as a Hölder 1/2-continuous map on $\Delta \cup \gamma$.*

We begin the proof by the following well-known assertion (see, for instance, [3]).

Lemma 4.2. *Let ϕ be a positive subharmonic function in Δ such that the measures $\mu_r(e^{i\theta}) := \phi(re^{i\theta})d\theta$ converge in the weak-star topology to a measure μ on $\partial\Delta$ as $r \to 1$. Suppose that μ vanishes on an open arc $\gamma \subset \partial\Delta$. Then for every compact subset $K \subset \Delta \cup \gamma$ there exists a constant $C > 0$ such that $\phi(\zeta) \leq C(1 - |\zeta|)$ for any $\zeta \in K \cup \Delta$.*

Now fix a point $a \in \gamma$, a constant $\delta > 0$ small enough so that the intersection $\gamma \cap (a + \delta\bar{\Delta})$ is compact in γ; we denote by Ω_δ the intersection $\Delta \cap (a + \delta\Delta)$. By Lemma 4.2, there exists a constant $C > 0$ such that, for any ζ in Ω_δ, we have

$$\rho \circ f(\zeta) \leq C(1 - |\zeta|). \tag{4.1}$$

Let $(\zeta_k)_k$ be a sequence of points in Δ converging to a with $\lim_{k\to\infty} f(\zeta_k) = p$. By assumption, the function ρ is strictly J-plurisubharmonic in a neighborhood U of p; hence there is a constant $\varepsilon > 0$ such that the function $\rho - \varepsilon|z|^2$ is J-plurisubharmonic on U.

Lemma 4.3. *There exists a constant $A > 0$ with the following property : If ζ is an arbitrary point of $\Omega_{\delta/2}$ such that $f(\zeta)$ is in $G \cap z^{-1}(\mathbb{B})$, then $|||df_\zeta||| \leq A(1 - |\zeta|)^{-1/2}$.*

Proof of Lemma 4.3. Set $d = 1 - |\zeta|$; then the disc $\zeta + d\Delta$ is contained in Ω_δ. Define the domain $G_d = \{w \in G : \rho(w) < 2Cd\}$. Then it follows by (4.1) that the image $f(\zeta + d\Delta)$ is contained in G_d, where the J-plurisubharmonic function $u_d = \rho - 2Cd$ is negative. Moreover we have the following lower estimates on the Kobayashi-Royden infinitesimal pseudometric (a rather technical proof is given in the Appendix) :

Proposition 4.4. *Let D be a domain in an almost complex manifold (M, J), let $p \in \bar{D}$, let U be a neighborhood of p in M (not necessarily contained in D) and let $z : U \to B$ be a normalized coordinate diffeomorphism introduced in Lemma 2.1. Let u be a $\mathcal{C}^2$ function on D, negative and J-plurisubharmonic on D. We assume that $-L \leq u < 0$ on $D \cap U$ and that $u - c|z|^2$ is J-plurisubharmonic on $D \cap U$, where c and L are positive constants. Then there exists a neighborhood $U' \subset U$ of p depending on c and $\|J\|_{C^2(U)}$, a positive constant c', depending only on c and L, such that we have the following estimate:*

$$K_{(D,J)}(q, v) \geq c'\|v\|/|u(q)|^{1/2}$$

for every $q \in D \cap U'$ and every $v \in T_qM$.

Hence there exists a positive constant M (independent of d) such that $K_{(G_d,J)}(w, \eta) \geq M|\eta||u_d(w)|^{-1/2}$, for any w in $G \cap z^{-1}(\mathbb{B})$ and any $\eta \in T_w\Omega$. On another hand, we have $K_{\zeta+d\Delta}(\zeta, \tau) = |\tau|/d$ for any τ in $T_\zeta\Delta$ indentified with $\mathbb{C}$. By the decreasing property of the Kobayashi-Royden metric, for any τ we have

$$M\|df_\zeta(\tau)\|\ |u_d(f(\zeta))|^{-1/2} \leq K_{(G_d,J)}(f(\zeta), df_\zeta(\tau)) \leq K_{\zeta+d\Delta}(\zeta,\tau) = |\tau|/d.$$

Therefore, $|||df_\zeta||| \leq M^{-1}|u_d(f(\zeta))|^{1/2}/d$. As $-2Cd \leq u_d(f(\zeta)) < 0$, this implies the desired statement in Lemma 4.3 with $A = M^{-1}(2C)^{1/2}$. □

Proof of Proposition 4.1. Lemma 4.3 implies that f extends as a 1/2-Hölder map to a neighborhood of the point a in view of an integration argument inspired by the classical Hardy-Littlewood theorem. This proves Proposition 4.1. □

4.2. **Reflection principle and regularity of analytic discs.** In the previous subsection we proved that a J-holomorphic disc attached to a smooth totally real submanifold is 1/2-Hölderian up to the boundary. This allows to use the reflection principle for pseudoholomorphic curves. Similar ideas have been used by E.Chirka [8] and S.Ivashkovich-V.Shevchischin [20]. For reader's convenience we present the argument due to E. Chirka.

Proposition 4.5. *Let E be an n-dimensional smooth totally real submanifold in an almost complex manifold (M,J). For any $p \in E$ there exists a neighborhood U of p and a smooth coordinate diffeomorphism $z : U \to \mathbb{B}$ such that $z(E) = \mathbb{R}^n$ and $z_*(J)_{|\mathbb{R}^n} = J_{st}$. Moreover, the condition of $z_*(J)$-holomorphy for a disc f may be written in the form*

$$\bar\partial f + A(f)\overline{\partial f} = 0$$

where the smooth matrix function $A(z)$ vanishes with infinite order on $\mathbb{R}^n$.

Proof. After a complex linear change of coordinates we may assume that $J = J_{st} + O(|z|)$ and E is given by $x + ih(x)$ where $x \in \mathbb{R}^n$ and $dh(0) = 0$. If Φ is the local diffeomorphism $x \mapsto x$, $y \mapsto y - h(x)$ then $\Phi(E) = \mathbb{R}^n$ and the direct image of J by Φ, still denoted by J, keeps the form $J_{st} + O(|z|)$. Then J has a basis of $(1,0)$-forms given in the coordinates z by $dz^j + \sum_k a_{jk} d\bar z^k$; using the matrix notation we write it in the form $\omega = dz + A(z)d\bar z$ where the matrix function $A(z)$ vanishes at the origin. Writing $\omega = (I+A)dx + i(I-A)dy$ where I denotes the identity matrix, we can take as a basis of $(1,0)$ forms : $\omega' = dx + i(I+A)^{-1}(I-A)dy = dx + iBdy$. Here the matrix function B satisfies $B(0) = I$. Since B is smooth, its restriction $B_{|\mathbb{R}^n}$ on $\mathbb{R}^n$ admits a smooth extension $\hat B$ on the unit ball such that $\hat B - B_{|\mathbb{R}^n} = O(|y|^k)$ for any positive integer k. Consider the diffeomorphism $z^* = x + i\hat B(z)y$. In the z^*-coordinates the submanifold E still coincides with $\mathbb{R}^n$ and $\omega' = dx + iBdy = dz^* + i(B - \hat B)dy - i(d\hat B)y = dz^* + \alpha$, where the coefficients of the form α vanish with infinite order on $\mathbb{R}^n$. Therefore there is a basis of $(1,0)$-forms (with respect to the image of J under the coordinate diffeomorphism $z \mapsto z^*$) of the form $dz^* + A(z^*)d\bar z^*$, where A vanishes with infinite order on $\mathbb{R}^n$. □

Now we are able to prove the main result of this section. We denote by Δ^+ the upper half disc $\Delta^+ = \{\zeta \in \mathbb{C} : Im(\zeta) > 0\}$.

Proposition 4.6. *Let E be a smooth totally real n-dimensional submanifold in a real $2n$-dimensional almost complex manifold (M,J) and let $f : \Delta^+ \to M$ be a J-holomorphic map. If the cluster set $C(f,]-1,1[)$ is (compactly) contained in E, then f is of class $\mathcal{C}^\infty$ on $\Delta^+ \cup]-1,1[$.*

Proof. We know that f is 1/2-Hölder continuous on $\Delta^+ \cup]-1,1[$ by Proposition 4.1 (see the beginning of Subsection 4.1). Moreover, it is easy to see that the Hölder constant depends on $\|f\|_\infty$, on the Levi form of the positive strictly J-plurisubharmonic defining function of E and on the $\mathcal{C}^2$-norm of J. Fix a point $a \in]-1,1[$. We may assume that $g(a) := z \circ f(a) = 0$. It is enough to show that the map g is of class $\mathcal{C}^\infty$ on $\Delta^+ \cap]-1,1[$ in a neighborhood of a. Consider the map $\hat g$ equal to g

on $\Delta^+ \cup]-1,1[$ and defined by $\hat{g}(\zeta) = \overline{g(\bar{\zeta})}$ for $\zeta \in \Delta^-$. Then $\hat{g}$ is continuous on Δ. We write the $z_*(J)$-holomorphy condition for g on Δ^+ in the form $\bar{\partial} g + q(g)\overline{\partial g} = 0$ where q is a smooth matrix function satisfying $\|q\|_{\mathcal{C}^k} << 1$ for any k. Then for $\zeta \in \Delta^-$ we have $\bar{\partial}\hat{g}(\zeta) + \overline{q(g(\bar{\zeta}))}\ \overline{\partial\hat{g}(\zeta)} = 0$. This means that $\hat{g}$ satisfies on Δ an elliptic equation of the form $\bar{\partial}\hat{g} + \phi(\cdot)\overline{\partial\hat{g}} = 0$ where ϕ is defined by $\phi(\zeta) = q(g(\zeta))$ for $\zeta \in \Delta^+ \cup]-1,1[$ and $\phi(\zeta) = \overline{q(g(\bar{\zeta}))}$ for $\zeta \in \Delta^-$. Since q vanishes on $\mathbb{R}^n$ with infinite order and g is $1/2$-Hölder continuous up to $]-1,1[$, the matrix function ϕ is smooth on Δ. The result now follows by the well-known results on elliptic regularity (see, for instance, [22, 28]) since $\hat{g}$ is necessarily $\mathcal{C}^\infty$-smooth on Δ. $\square$

Moreover, it follows from these results that for any k and for any $r > 0$ there exists a constant such that $\|\hat{g}\|_{\mathcal{C}^k((1-r)\Delta)} \leq C\|\hat{g}\|_{\mathcal{C}^0((1-r)\Delta)}$. This estimate will be used in the following section to study the boundary regularity of pseudoholomorphic maps. Since the initial disc f is obtained from $\hat{g}$ by a diffeomorphism depending on E and J only, we obtain the following quantitative version of the previous statement.

Proposition 4.7. *Let E be a totally real n-dimensional submanifold in an (complex) n-dimensional almost complex manifold (M, J) and let $f : \Delta^+ \to \mathbb{C}^n$ be a J-holomorphic map. Assume that the cluster set $C(f,]-1,1[)$ is compactly contained in E. Then given $r > 2$ there exists a constant $C > 0$ depending on $\|f\|_\infty$ and on the $\mathcal{C}^r$ norm of the defining functions of E such that*

$$\|f\|_{\mathcal{C}^r(\Delta^+ \cup]-1,1[)} \leq C\|J\|_{\mathcal{C}^r}. \tag{4.2}$$

In the next Section we apply these results to the study of the bounadary regularity of pseudoholomorphic maps of wedges with totally real edges.

5. Behavior of pseudoholomorphic maps near totally real submanifolds

Let Ω be a domain in an almost complex manifold (M, J) and $E \subset \Omega$ be a smooth n-dimensional totally real submanifold defined as the set of common zeros of the functions r_j, $j = 1, ..., n$ smooth on Ω. We suppose that $\bar{\partial}_J r_1 \wedge ... \wedge \bar{\partial}_J r_n \neq 0$ on Ω. Consider the "wedge" $W(\Omega, E) = \{z \in \Omega : r_j(z) < 0, j = 1, ..., n\}$ with "edge" E. For $\delta > 0$ we denote by $W_\delta(\Omega, E)$ the "shrinked" wedge $\{z \in \Omega : r_j(z) - \delta\sum_{k\neq j} r_k < 0, j = 1, ..., n\}$. The main goal of this Section is to prove the following

Proposition 5.1. *Let $W(\Omega, E)$ be a wedge in $\Omega \subset (M, J)$ with a totally real n-dimensional edge E of class $\mathcal{C}^\infty$ and let $f : W(\Omega, E) \to (M', J')$ be a (J, J')-holomorphic map. Suppose that the cluster set $C(f, E)$ is (compactly) contained in a $\mathcal{C}^\infty$ totally real submanifold E' of M'. Then for any $\delta > 0$ the map f extends to $W_\delta(\Omega, E) \cup E$ as a $\mathcal{C}^\infty$-map.*

In Section 4 we established this statement for a single J-holomorphic disc. The general case also relies on the ellipticity of the $\bar{\partial}$-operator. It requires an additional technique of attaching pseudoholomorphic discs to a totally real manifold which could be of independent interest.

5.1. **Almost complex perturbation of discs.** In this subsection we attach Bishop's discs to a totally real submanifold in an almost complex manifold. The following statement is an almost complex analogue of the well-known Pinchuk's construction [24] of a family of holomorphic discs attached to a totally real manifold.

Lemma 5.2. *For any $\delta > 0$ there exists a family of J-holomorphic discs $h(\tau, t) = h_t(\tau)$ smoothly depending on the parameter $t \in \mathbb{R}^{2n}$ such that $h_t(\partial\Delta^+) \subset E$, $h_t(\Delta) \subset W(\Omega, E)$, $W_\delta(\Omega, E) \subset \cup_t h_t(\Delta)$ and $C_1(1 - |\tau|) \leq dist(h_t(\tau), E) \leq C_2(1 - |\tau|)$ for any t and any $\tau \in \Delta^+$, with constants $C_j > 0$ independent of t.*

For $\alpha > 1$, noninteger, we denote by $\mathcal{C}^\alpha(\bar\Delta)$ the Banach space of functions of class $\mathcal{C}^\alpha$ on $\bar\Delta$ and by $\mathcal{A}^\alpha$ the Banach subspace of $\mathcal{C}^\alpha(\bar\Delta)$ of functions holomorphic on Δ.

First we consider the situation where $E = \{r := (r_1, \dots, r_n) = 0\}$ is a smooth totally real submanifold in $\mathbb{C}^n$. Let J_λ be an almost complex deformation of the standard structure J_{st} that is a one-parameter family of almost complex structures so that $J_0 = J_{st}$. We recall that for λ small enough the (J_{st}, J_λ)-holomorphy condition for a map $f : \Delta \to \mathbb{C}^n$ may be written in the form

$$\bar\partial_{J_\lambda} f = \bar\partial f + q(\lambda, f)\overline{\partial f} = 0 \tag{5.1}$$

where q is a smooth matrix satisfying $q(0, \cdot) \equiv 0$, uniquely determined by J_λ ([28]).

A disc $f \in (\mathcal{C}^\alpha(\bar\Delta))^n$ is attached to E and is J_λ-holomorphic if and only if it satisfies the following nonlinear boundary Riemann-Hilbert type problem :

$$\begin{cases} r(f(\zeta)) = 0, & \zeta \in \partial\Delta \\ \bar\partial_{J_\lambda} f(\zeta) = 0, & \zeta \in \Delta. \end{cases}$$

Let $f^0 \in (\mathcal{A}^\alpha)^n$ be a disc attached to E and let $\mathcal{U}$ be a neighborhood of $(f^0, 0)$ in the space $(\mathcal{C}^\alpha(\bar\Delta))^n \times \mathbb{R}$. Given (f, λ) in $\mathcal{U}$ define the maps $v_f : \zeta \in \partial\Delta \mapsto r(f(\zeta))$ and

$$\begin{aligned} u : \mathcal{U} &\to (\mathcal{C}^\alpha(\partial\Delta))^n \times \mathcal{C}^{\alpha-1}(\Delta) \\ (f, \lambda) &\mapsto (v_f, \bar\partial_{J_\lambda} f). \end{aligned}$$

Denote by X the Banach space $(\mathcal{C}^\alpha(\bar\Delta))^n$. Since r is of class $\mathcal{C}^\infty$, the map u is smooth and the tangent map $D_X u(f^0, 0)$ (we consider the derivative with respect to the space X) is a linear map from X to $(\mathcal{C}^\alpha(\partial\Delta))^n \times \mathcal{C}^{\alpha-1}(\Delta)$, defined for every $h \in X$ by

$$D_X u(f^0, 0)(h) = \begin{pmatrix} 0 & 2Re[Gh] \\ 0 & \bar\partial_{J_0} h \end{pmatrix},$$

where for $\zeta \in \partial\Delta$

$$G(\zeta) = \begin{pmatrix} \frac{\partial r_1}{\partial z^1}(f^0(\zeta)) & \cdots & \frac{\partial r_1}{\partial z^n}(f^0(\zeta)) \\ \cdots & \cdots & \cdots \\ \frac{\partial r_n}{\partial z^1}(f^0(\zeta)) & \cdots & \frac{\partial r_n}{\partial z^n}(f^0(\zeta)) \end{pmatrix}$$

(see [16]).

Lemma 5.3. *Assume that for some $\alpha > 1$ the linear map from $(\mathcal{A}^\alpha)^n$ to $(\mathcal{C}^{\alpha-1}(\Delta))^n$ given by $h \mapsto 2Re[Gh]$ is surjective and has a d-dimensional kernel. Then there exist $\delta_0, \lambda_0 > 0$ such that for every $0 \le \lambda \le \lambda_0$, the set of J_λ-holomorphic discs f attached to E and such that $\|f - f^0\|_\alpha \le \delta_0$ forms a smooth d-dimensional submanifold $\mathcal{A}_\lambda$ in the Banach space $(C^\alpha(\bar\Delta))^n$.*

Proof of Lemma 5.3. According to the implicit function Theorem, the proof of Lemma 5.3 reduces to the proof of the surjectivity of $D_X u$. It follows by classical one-variable results on the resolution of the $\bar\partial$-problem in the unit disc that the linear map from X to $\mathcal{C}^{\alpha-1}(\Delta)$ given by $h \mapsto \bar\partial h$ is surjective. More precisely, given $g \in \mathcal{C}^{\alpha-1}(\Delta)$ consider the Cauchy transform

$$T_\Delta(g) : \tau \in \partial\Delta \mapsto \frac{1}{2i\pi} \int\int_\Delta \frac{g(\zeta)}{\zeta - \tau} d\zeta \wedge d\bar\zeta.$$

For every function $g \in C^{\alpha-1}(\Delta)$ the solutions $h \in X$ of the equation $\bar\partial h = g$ have the form $h = h_0 + T_\Delta(g)$ where h_0 is an arbitrary function in $(\mathcal{A}^\alpha)^n$. Consider the equation

$$D_X u(f^0, 0)(h) = \begin{pmatrix} 0 & g_1 \\ 0 & g_2 \end{pmatrix} \tag{5.2}$$

where (g_1, g_2) is a vector-valued function with components $g_1 \in \mathcal{C}^{\alpha-1}(\partial\Delta)$ and $g_2 \in \mathcal{C}^{\alpha-1}(\Delta)$. Solving the $\bar\partial$-equation for the second component, we reduce equation (5.2) to

$$2Re[G(\zeta)h_0(\zeta)] = g_1 - 2Re[G(\zeta)T_\Delta(g_2)(\zeta)]$$

with respect to $h_0 \in (\mathcal{A}^\alpha)^n$. The surjectivity of the map $h_0 \mapsto 2Re[Gh_0]$ gives the result. $\square$

Proof of Lemma 5.2. We proceed in three steps. *Step 1. Filling the polydisc.* Consider the n-dimensional real torus $\mathbb{T}^n = \partial\Delta \times ... \times \partial\Delta$ in $\mathbb{C}^n$ and the linear disc $f^0(\zeta) = (\zeta, ..., \zeta)$, $\zeta \in \Delta$ attached to $\mathbb{T}^n$. In that case, a disc h^0 is in the kernel of $h \mapsto 2Re[Gh]$ if and only if every component h^0_k of h^0 satisfies on $\partial\Delta$ the condition $h^0_k + \zeta^2\overline{h^0_k} = 0$. Considering the Fourier expansion of h_k on $\partial\Delta$ (recall that h_k is holomorphic on Δ) and identifying the coefficients, we obtain that the map $h \mapsto 2Re[Gh]$ from $(\mathcal{A}^\alpha)^n$ to $(C^{\alpha-1}(\Delta))^n$ is surjective and has a $3n$-dimensional kernel. By Lemma 5.3 if J_λ is an almost complex structure close enough to J_{st} in a neighborhood of the closure of the polydisc Δ^n, there is a $3n$-parameters family of J_λ-holomorphic discs attached to $\mathbb{T}^n$. These J_λ-holomorphic discs fill the intersection of a sufficiently small neighborhood of the point $(1, ..., 1)$ with Δ^n.

Step 2. Isotropic dilations. Consider a smooth totally real submanifold E in an almost complex manifold (M, J). Fixing local coordinates, we may assume that E is a submanifold in a neighborhood of the origin in $\mathbb{C}^n$, $J = J_{st} + 0(|z|)$ and E is defined by the equations $y = \phi(x)$, where $\nabla\phi(0) = 0$. For every $\varepsilon > 0$, consider the isotropic dilations $\Lambda_\varepsilon : z \mapsto z' = \varepsilon^{-1}z$. Then $J_\varepsilon := \Lambda_\varepsilon(J) \to J_{st}$ as $\varepsilon \to 0$. In the z'-coordinates E is defined by the equations $y' = \psi(x', \varepsilon) := \varepsilon^{-1}\phi(\varepsilon x')$ and $\psi \to 0$ as $\varepsilon \to 0$. Consider the local diffeomorphism $\Phi_\varepsilon : z' = x' + iy' \mapsto z'' = x' + i(y' - \psi(x', \varepsilon))$. Then in new coordinates (we omit the primes) E coincides with a neighborhood of the origin in $\mathbb{R}^n = \{y = 0\}$ and $\hat{J}_\varepsilon := (\Phi_\varepsilon)_*(J_\varepsilon) \to J_{st}$ as $\varepsilon \to 0$. Furthermore, applying a fractional-linear transformation of $\mathbb{C}^n$, biholomorphic with respect to J_{st}, we may assume that E is a neighborhood of the point $(1, ..., 1)$ on the torus $\mathbb{T}^n$ and the almost complex structure J_ε is a small deformation of the standard structure. By Step 1, we may fill a neigborhood of the point $(1, ..., 1)$ in the polydisc Δ^n by J_ε-holomorphic discs (for ε small enough) which are small perturbations of the disc $\zeta \mapsto (\zeta, ..., \zeta)$. Returning to the initial coordinates, we obtain a family of J-holomorphic discs attached to E along a fixed arc (say, the upper semi-circle $\partial\Delta^+$) and filling the intersection of a neighborhhod of the origin with the wedge $\{y - \phi(x) < 0\}$.

Step3. Let now $W(\Omega, E) = \{r_j < 0, j = 1, ..., n\}$ be a wedge with edge E; we assume that $0 \in E$ and $J(0) = J_{st}$. We may assume that $E = \{y = \phi(x)\}$, $\nabla\phi(0) = 0$, since the linear part of every r_j at the origin is equal to y_j. So shrinking Ω if necessary, we obtain that for any $\delta > 0$ the wedge $W_\delta(\Omega, E) = \{z \in \Omega : r_j(z) - \delta\sum_{k\neq j} r_k(z) < 0, j = 1, ..., n\}$ is contained in the wedge $\{z \in \Omega : y - \phi(x) < 0\}$. By Step 2 there is a family of J-holomorphic discs attached to E along the upper semi-circle and filling the wedge $W_\delta(\Omega, E)$. These discs are smooth up to the boundary and smoothly depend on the parameters. $\square$

5.2. **Uniform estimates of derivatives.** Now we prove Proposition 5.1. Let $(h_t)_t$ be the family of J-holomorphic discs, smoothly depending on the parameter $t \in \mathbb{R}^{2n}$, defined in Lemma 5.2. It follows from Lemma 4.3, applied to the holomorphic disc $f \circ h_t$, uniformly with respect to t, that there is a constant C such that $|||df(z)||| \leq Cdist(z,E)^{-1/2}$ for any $z \in W_\delta(\Omega, E)$. This implies that f extends as a Hölder 1/2-continuous map on $W_\delta(\Omega, E) \cup E$.

It follows now from Proposition 4.6 that every composition $f \circ h_t$ is smooth up to $\partial\Delta^+$. Moreover, since f is continuous up to E, the estimate (4.2) shows that in our case the $\mathcal{C}^k$ norm of the discs $f \circ h_t$ are uniformly bounded, for any k. Recall the separate smoothness principle (Proposition 3.1, [29]):

Proposition 5.4. *Let F_j, $1 \leq j \leq n$, be $\mathcal{C}^\alpha$ ($\alpha > 1$ noninteger) smooth foliations in a domain $\Omega \subset \mathbb{R}^n$ such that for every point $p \in \Omega$ the tangent vectors to the curves $\gamma_j \in F_j$ passing through p are linearly independent. Let f be a function on Ω such that the restrictions $f_{|\gamma_j}$, $1 \leq j \leq n$, are of class $\mathcal{C}^{\alpha-1}$ and are uniformly bounded in the $\mathcal{C}^{\alpha-1}$ norm. Then f is of class $\mathcal{C}^{\alpha-1}$.*

Using Lemma 5.2 we construct n transversal foliations of E by boundaries of Bishop's discs. Since the restriction of f on every such curve satisfies the hypothesis of Proposition 5.4, f is smooth up to E. This proves Proposition 5.1. $\square$

6. Lifts of biholomorphisms to the cotangent bundle

We first recall the notion of conormal bundle of a real submanifold in $\mathbb{C}^n$ with the standard structure ([30]). Let $T^*(\mathbb{C}^n)$ be the real cotangent bundle of $\mathbb{C}^n$, identified with the bundle $T^*_{(1,0)}(\mathbb{C}^n)$ of complex (1,0) forms and let $\pi : T^*(\mathbb{C}^n) \to \mathbb{C}^n$ be the natural projection. In the canonical complex coordinates (z,t) on $T^*_{(1,0)}(\mathbb{C}^n)$ an element of the fiber at a point $z \in \mathbb{C}^n$ is a (1,0) form $\omega = \sum_j t_j dz^j$. Let N be a real smooth generic submanifold in $\mathbb{C}^n$. The conormal bundle $\Sigma(N)$ of N is a real subbundle of $T^*_{(1,0)}(\mathbb{C}^n)$ defined by the condition $\Sigma(N) = \{\phi \in T^*_{(1,0)}(\mathbb{C}^n) : Re\,\phi|T^{(1,0)}_z(N) = 0, z \in N\}$. By $T^{(1,0)}_z(N)$ we mean here the real tangent space of N at z considered as a subspace in $T^{1,0}_z(\mathbb{C}^n)$ after the canonical identification of the tangent bundles $T(\mathbb{C}^n)$ and $T^{(1,0)}(\mathbb{C}^n)$.

Let $\rho_1, \dots, \rho_d$ be local defining functions of N. Then the forms $\partial\rho_1, \dots, \partial\rho_d$ form a basis in $\Sigma_z(N)$ and every section ϕ of the bundle $\Sigma(N)$ has the form $\phi = \sum_{j=1}^d c_j \partial\rho_j$, $c_1, \dots, c_d \in \mathbb{R}$. We will use the following statement (see [30]) :

Lemma 6.1. *Let Γ be a $\mathcal{C}^2$ real hypersurface in $\mathbb{C}^n$. The conormal bundle $\Sigma(\Gamma)$ (except the zero section) is a totally real submanifold of dimension $2n$ in $T^*_{(1,0)}(\mathbb{C}^n)$ if and only if the Levi form of Γ is nondegenerate.*

The conormal bundle notion can be easily extended to the case of an almost complex manifold. Let $i : T^*(M) \to T^*_{(1,0)}(M,J)$ be the canonical identification. Let D be a smoothly relatively compact domain in M with boundary Γ. The conormal bundle $\Sigma_J(\Gamma)$ of Γ is the real subbundle of $T^*_{(1,0)}(M,J)$ defined by $\Sigma_J(\Gamma) = \{\phi \in T^*_{(1,0)}(M,J) : Re\,\phi|T^J_z(\Gamma) = 0, z \in \Gamma\}$. As above, by $T^J_z(\Gamma)$ we mean the real tangent space of Γ at z viewed as a real subspace of the (1,0) (with respect to J) tangent space of M at z.

This notion is invariant with respect to biholomorphisms. More precisely, if $f : (D,J) \to (D',J')$ is a biholomorphic map $\mathcal{C}^1$-smooth up to ∂D, then its *cotangent map* defined by $\tilde{f} := (f, {}^t df^{-1})$ is continuous up to $\Sigma_J(\partial D)$ and $\tilde{f}(\Sigma_J(\partial D)) = \Sigma_{J'}(\partial D')$. To apply the results of the previous sections, we define an almost complex structure on the cotangent bundle $T^*(M)$ of an almost complex manifold such that the cotangent map of a biholomorphism is biholomorphic with respect to this structure. For reader's convenience we recall the explicit construction of this almost complex structure $\tilde{J}$ (ie. the proof of the following Proposition), following [32], in Appendix 2.

Proposition 6.2. *Let (M, J) be an almost complex manifold. There exists an almost complex structure $\tilde{J}$ on T^*M with the following properties :*

(i) If f is a biholomorphism between (M, J) and (M', J') then the cotangent map $\tilde{f}$ is a biholomorphism between $(T^*M, \tilde{J})$ and $(T^*M', \tilde{J}')$.

(ii) If $(J_\varepsilon)_\varepsilon$ is a small deformation of the standard structure on $\mathbb{C}^n$ then $\tilde{J}_\varepsilon \to J_{st}$ as $\varepsilon \to 0$ in the $\mathcal{C}^k$-norm on $T^*\mathbb{C}^n$ (for any k).

Consider now a smooth relatively compact strictly pseudoconvex domain D in an almost complex manifold (M, J) of real dimension four. We have the following

Lemma 6.3. *The conormal bundle of ∂D (outside the zero section) is a totally real submanifold in $(T^*(M), \tilde{J})$.*

Proof of Lemma 6.3. According to Section 2 we may choose local coordinates near a boundary point p such that ∂D is given by the equation $Rez^2 + ReK(z) + H(z) + o(|z|^2) = 0$ and $H(z^1, 0)$ is a positive definite hermitian form on $\mathbb{C}$; in these coordinates the matrix J is diagonal and $J(0) = J_{st}$. After the non-isotropic dilation $(z^1, z^2) \mapsto (\varepsilon^{-1/2}z^1, \varepsilon^{-1}z^2)$ the hypersurface is defined by the equation $Rez^2 + \varepsilon^{-1}K(\varepsilon^{1/2}z^1, \varepsilon z^2) + H(\varepsilon^{1/2}z^1, \varepsilon z^2) + \varepsilon^{-1}o(|(\varepsilon^{1/2}z^1, \varepsilon z^2)|^2) = 0$ and the dilated structure, denoted by J_ε, tends (together with all derivatives of any order) to the standard structure. The hypersurface ∂D tends to the strictly J_{st}-pseudoconvex hypersurface $\Gamma_0 = \{Rez^2 + K(z^1, 0) + H(z^1, 0) = 0\}$. It follows from Proposition 6.2 (ii) that $\tilde{J}_\varepsilon$ tends to the standard complex structure on $T^*(\mathbb{C}^n)$ as $\varepsilon \to 0$. Since the conormal bundle of Γ_0 (with respect to J_{st}) is totally real with respect to the standard structure on $T^*(\mathbb{C}^n)$, the same holds for $\Sigma_J(\partial D)$ in a small neighborhood of p, by continuity (for ε small enough). □

If $f : (D, J) \to (D', J')$ is a biholomorphism between two strictly pseudoconvex domains, of class $\mathcal{C}^1$ on $\bar{D}$, then its cotangent lift extends continuously on $\Sigma_J(\partial D)$ and $f(\Sigma_J(\partial D) \subset \Sigma_{J'}(\partial D')$. In view of Proposition 5.1 this proves Fefferman's theorem under the additional assumption of $\mathcal{C}^1$-smoothness of f up to the boundary :

Proposition 6.4. *Let D and D' be smooth relatively compact strictly pseudoconvex domains in (real) four dimensional almost complex manifolds (M, J) and (M', J'). Consider a $\mathcal{C}^\infty$- diffeomorphism $f : (D, J) \to (D', J')$ which is a $\mathcal{C}^1$-diffeomorphism between $\bar{D}$ and $\bar{D}'$. Suppose that the direct image $f_*(J)$ extends $\mathcal{C}^\infty$-smoothly on $\bar{D}'$. Then f is a $\mathcal{C}^\infty$-diffeomorphism between $\bar{D}$ and $\bar{D}'$.*

This statement just follows by the reflection principle of Sections 4 and 5. In order to get rid of the $\mathcal{C}^1$-assumption we will use the estimates on the Kobayashi-Royden metric and the scaling method.

7. Scaling on almost complex manifolds

Our goal now is to prove Fefferman's mapping theorem without the assumption of $\mathcal{C}^1$-smoothness of f up to the boundary. This equires an application of the estimates of the Kobayashi-Royden metric given in Section 3 and the scaling method due to S.Pinchuk; we adapt this to the almost complex case.

In Section 3 we reduced the problem to the following local situation. Let D and D' be domains in $\mathbb{C}^2$, Γ and Γ' be open $\mathcal{C}^\infty$-smooth pieces of their boundaries, containing the origin. We assume that an almost complex structure J is defined and $\mathcal{C}^\infty$-smooth in a neighborhood of the closure $\bar{D}$, $J(0) = J_{st}$ and J has a diagonal form in a neighborhood of the origin: $J(z) = diag(a_{11}(z), a_{22}(z))$. Similarly, we assume that J' is diagonal in a neighborhood of the origin, $J'(z) = diag(a'_{11}(z), a'_{22}(z))$

and $J'(0) = J_{st}$. The hypersurface Γ (resp. Γ') is supposed to be strictly J-pseudoconvex (resp. strictly J'-pseudoconvex). Finally, we assume that $f : D \to D'$ is a (J, J')-biholomorphic map, 1/2-Hölder homeomorphism between $D \cup \Gamma$ and $D' \cup \Gamma'$, such that $f(\Gamma) = \Gamma'$ and $f(0) = 0$. Finally according to Section 2, Γ is defined in a neighborhood of the origin by the equation $\rho(z) = 0$ where $\rho(z) = 2Rez^2 + 2ReK(z) + H(z) + o(|z|^2)$ and $K(z) = \sum K_{\mu\nu} z^{\mu\nu}$, $H(z) = \sum h_{\mu\nu} z^\mu \bar{z}^\nu$, $k_{\mu\nu} = k_{\nu\mu}$, $h_{\mu\nu} = \bar{h}_{\nu\mu}$. The crucial point is that $H(z^1, 0)$ is a positive hermitian form on $\mathbb{C}$, meaning that in these coordinates Γ is strictly pseudoconvex at the origin with respect to the standard structure of $\mathbb{C}^2$ (see Lemma 2.13 for the proof). Of course, Γ' admits a similar local representation. In what follows we assume that we are in this setting.

Let (p^k) be a sequence of points in D converging to 0 and let $\Sigma := \{z \in \mathbb{C}^2 : 2Rez^2 + 2ReK(z^1, 0) + H(z^1, 0) < 0\}$, $\Sigma' := \{z \in \mathbb{C}^2 : 2Rez^2 + 2ReK'(z^1, 0) + H'(z^1, 0) < 0\}$. The scaling procedure associates with the pair $(f, (p^k)_k)$ a biholomorphism ϕ (with respect to the standard structure J_{st}) between Σ and Σ'. Since ϕ is obtained as a limit of a sequence of biholomorphic maps conjugated with f, some of their properties are related and this can be used to study boundary properties of f and to prove that its cotangent lift is continuous up to the conormal bundle $\Sigma(\partial D)$.

7.1. **Fixing suitable local coordinates and dilations.** For any boundary point $t \in \partial D$ we consider the change of variables α^t defined by

$$(z^1)^* = \frac{\partial \rho}{\partial \bar{z}^2}(t)(z^1 - t^1) - \frac{\partial \rho}{\partial \bar{z}^1}(t)(z^2 - t^2),\ (z^2)^* = \sum_{j=1}^{2} \frac{\partial \rho}{\partial z^j}(t)(z^j - t^j).$$

Then α^t maps t to 0. The real normal at 0 to Γ is mapped by α^t to the line $\{z^1 = 0, y_2 = 0\}$. For every k, we denote by t^k the projection of p^k onto ∂D and by α^k the change of variables α^t with $t = t^k$. Set $\delta_k = dist(p^k, \Gamma)$. Then $\alpha^k(p^k) = (0, -\delta_k)$ and $\alpha^k(D) = \{2Rez^2 + O(|z|^2) < 0\}$ near the origin. Since the sequence $(\alpha^k)_k$ converges to the identity map, the sequence $(\alpha^k)_*(J)$ of almost complex structures tends to J as $k \to \infty$. Moreover there is a sequence (L^k) of linear automorphisms of $\mathbb{R}^4$ such that $(L^k \circ \alpha^k)_*(J)(0) = J_{st}$. Then $(L^k \circ \alpha^k)(p^k) = (o(\delta_k), -\delta'_k)$ with $\delta'_k \sim \delta_k$ and $(L^k \circ \alpha^k)(D) = \{Re(z^2 + \tau_k z^1) + O(|z|^2) < 0\}$ near the origin, with $\tau_k = o(1)$. Hence there is sequence (M^k) of $\mathbb{C}$-linear transformations of $\mathbb{C}^2$, converging to the identity, such that $(T^k := M^k \circ L^k \circ \alpha^k)$ is a sequence of linear transformations converging to the identity, and $D^k := T^k(D)$ is defined near the origin by $D^k = \{\rho_k(z) = Rez^2 + O(|z|^2) < 0\}$. Finally $\tilde{p}_k = T^k(p^k) = (o(\delta_k), \delta''_k + io(\delta_k))$ with $\delta''_k \sim \delta_k$. We also denote by $\Gamma^k = \{\rho_k = 0\}$ the image of Γ under T^k. Furthermore, the sequence of almost complex structures $(J_k := (T^k)_*(J))$ converges to J as $k \to \infty$ and $J_k(0) = J_{st}$.

We proceed quite similarly for the target domain D'. For $s \in \Gamma'$ we define the transformation β^s by

$$(z^1)^* = \frac{\partial \rho'}{\partial \bar{z}^2}(s)(z^1 - s^1) - \frac{\partial \rho'}{\partial \bar{z}^1}(s)(z^2 - s^2), (z^2)^* = \sum_{j=1}^{2} \frac{\partial \rho'}{\partial z^j}(s)(z^j - s^j).$$

Let s^k be the projection of $q^k := f(p^k)$ onto Γ' and let β^k be the corresponding map β^s with $s = s^k$. The sequence (q^k) converges to $0 = f(0)$ so β^k tends to the identity. Considering linear transformations $(L')^k$ and $(M')^k$, we obtain a sequence (T'^k) of linear transformations converging to the identity and satisfying the following properties. The domain $(D^k)' := T'^k(D')$ is defined near the origin by $(D^k)' = \{\rho'_k(z) := Rez^2 + O(|z|^2) < 0\}$, $\Gamma'_k = \{\rho'_k = 0\}$ and $\tilde{q}_k = T'^k(q^k) = (o(\varepsilon_k), \varepsilon''_k + io(\varepsilon_k))$ with $\varepsilon''_k \sim \varepsilon_k$, where $\varepsilon_k = dist(q^k, \Gamma')$. The sequence of almost complex structures $(J'_k := (T'^k)_*(J'))$ converges to J' as $k \to \infty$ and $J'_k(0) = J_{st}$.

Finally, the map $f^k := T'^k \circ f \circ (T^k)^{-1}$ satisfies $f^k(\tilde{p}_k) = \tilde{q}_k$ and is a biholomorphism between the domains D^k and $(D')^k$ with respect to the almost complex structures J_k and J'_k.

Consider now the non isotropic dilations $\phi_k : (z^1, z^2) \mapsto (\delta_k^{1/2} z^1, \delta_k z^2)$ and $\psi_k(z^1, z^2) = (\varepsilon_k^{1/2} z^1, \varepsilon_k z^2)$ and set $\hat{f}^k = (\psi_k)^{-1} \circ f^k \circ \phi_k$. Then the map $\hat{f}^k$ is biholomorphic with respect to the almost complex structures $\hat{J}_k := ((\phi_k)^{-1})_*(J_k)$ and $\hat{J}'_k := (\psi_k^{-1})_*(J'_k)$. Moreover if $\hat{D}^k := \phi_k^{-1}(D^k)$ and $(\hat{D}')^k := \psi_k^{-1}((D')^k)$ then $\hat{D}^k = \{z \in \phi_k^{-1}(U) : \hat{\rho}_k(z) < 0\}$ where

$$\hat{\rho}_k(z) := \delta_k^{-1} \rho(\phi_k(z)) = 2Rez^2 + \delta_k^{-1}[2ReK(\delta_k^{1/2} z^1, \delta_k z^2) + H(\delta_k^{1/2} z^1, \delta_k z^2) + o(|(\delta_k^{1/2} z^1, \delta_k z^2)|^2).$$

and $(\hat{D}')^k = \{z \in \phi_k^{-1}(U) : \hat{\rho}'_k(z) < 0\}$ where

$$\hat{\rho}'_k(z) := \varepsilon_k^{-1} \rho'(\psi_k(z)) = 2Rez^2 + \varepsilon_k^{-1}[2ReK'(\varepsilon_k^{1/2} z^1, \varepsilon_k z^2) + H'(\varepsilon_k^{1/2} z^1, \varepsilon_k z^2) + o(|(\varepsilon_k^{1/2} z^1, \varepsilon_k z^2)|^2).$$

Since U is a fixed neighborhood of the origin, the pull-backs $\phi_k^{-1}(U)$ tend to $\mathbb{C}^2$ and the functions $\hat{\rho}_k$ tend to $\hat{\rho}(z) = 2Rez^2 + 2ReK(z^1, 0) + H(z^1, 0)$ in the $\mathcal{C}^2$ norm on any compact subset of $\mathbb{C}^2$. Similarly, since U' is a fixed neighborhood of the origin, the pull-backs $\psi_k^{-1}(U')$ tend to $\mathbb{C}^2$ and the functions $\hat{\rho}'_k$ tend to $\hat{\rho}'(z) = 2Rez^2 + 2ReK'(z^1, 0) + H'(z^1, 0)$ in the $\mathcal{C}^2$ norm on any compact subset of $\mathbb{C}^2$. If $\Sigma := \{z \in \mathbb{C}^2 : \hat{\rho}(z) < 0\}$ and $\Sigma' := \{z \in \mathbb{C}^2 : \hat{\rho}'(z) < 0\}$ then the sequence of points $\hat{p}^k = \phi_k^{-1}(\tilde{p}_k) \in \hat{D}^k$ converges to the point $(0, -1) \in \Sigma$ and the sequence of points $\hat{q}^k = \psi_k^{-1}(\tilde{q}^k) \in \hat{D'}^k$ converges to $(0, -1) \in \Sigma'$. Finally $\hat{f}^k(\hat{p}^k) = \hat{q}^k$.

7.2. **Convergence of the dilated families.** We begin with the following

Lemma 7.1. *The sequences $(\hat{J}'_k)$ and $(\hat{J}_k)$ of almost complex structures converge to the standard structure uniformly (with all partial derivatives of any order) on compact subsets of $\mathbb{C}^2$.*

Proof of Lemma 7.1. Denote by $a^k_{\nu\mu}(z)$ the elements of the matrix J_k. Since $J_k \to J$ and J is diagonal, we have $a^k_{\nu\mu} \to a_{\nu\mu}$ for $\nu = \mu$ and $a^k_{\nu\mu} \to 0$ for $\nu \neq \mu$. Moreover, since $J_k(0) = J_{st}$, $a^k_{\nu\mu}(0) = i$ for $\nu = \mu$ and $a_{\nu\mu}(0) = 0$ for $\nu \neq \mu$. The elements $\hat{a}^k_{\nu\mu}$ of the matrix $\hat{J}_k$ are given by: $\hat{a}^k_{\nu\mu}(z^1, z^2) = a^k_{\nu\mu}(\delta_k^{1/2} z^1, \delta_k z^2)$ for $\nu = \mu$, $\hat{a}^k_{12}(z^1, z^2) = \delta_k^{1/2} a(\delta_k^{1/2} z^1, \delta_k z^2)$ and $\hat{a}^k_{21}(z^1, z^2) = \delta_k^{-1/2} a^k_{21}(\delta_k^{1/2} z^1, \delta_k z^2)$. This implies the desired result. □

The next statement is crucial.

Proposition 7.2. *The sequence $(\hat{f}^k)$ (together with all derivatives) is a relatively compact family (with respect to the compact open topology) on Σ; every cluster point $\hat{f}$ is a biholomorphism (with respect to J_{st}) between Σ and Σ', satisfying $\hat{f}(0, -1) = (0, -1)$ and $(\partial \hat{f}^2 / \partial z^2)(0, -1) = 1$.*

Proof of Proposition 7.2. Step 1: convergence. Our proof is based on the method developped by F.Berteloot-G.Coeuré and F.Berteloot [5, 4]. Consider a domain $G \subset \mathbb{C}^2$ of the form $G = \{z \in W : \lambda(z) = 2Rez^2 + 2ReK(z) + H(z) + o(|z|^2) < 0\}$ where W is a neighborhood of the origin. We assume that an almost complex structure J is diagonal on W and that the hypersurface $\{\lambda = 0\}$ is strictly J-pseudoconvex at any point. Given $a \in \mathbb{C}^2$ and $\delta > 0$ denote by $Q(a, \delta)$ the non-isotropic ball $Q(a, \delta) = \{z : |z^1 - a_1| < \delta^{1/2}, |z^2 - a_2| < \delta\}$. Denote also by d_δ the non-isotropic dilation $d_\delta(z^1, z^2) = (\delta^{-1/2} z^1, \delta^{-1} z^2)$.

Lemma 7.3. *There exist positive constants δ_0, C, r satisfying the following property : for every $\delta \leq \delta_0$ and for every J-holomorphic disc $g : \Delta \to G$ such that $g(0) \in Q(0, \delta)$ we have the inclusion $g(r\Delta) \subset Q(0, C\delta)$.*

Proof of Lemma 7.3. Assume by contradiction that there exist positive sequences $\delta_k \to 0$, $C_k \to +\infty$, a sequence $\zeta_k \in \Delta$, $\zeta_k \to 0$ and a sequence $g_k : \Delta \to G$ of J-holomorphic discs such that $g_k(0) \in Q(0, \delta_k)$ and $g_k(\zeta_k) \notin Q(0, C_k\delta_k)$. Denote by d_k the dilations d_δ with $\delta = \delta_k$ and consider the composition $h_k = d_k \circ g_k$ defined on Δ. The dilated domains $G_k := d_k(G)$ are defined by $\{z \in d_k(W) : \lambda_k(z) := \delta_k^{-1}\lambda \circ d_k^{-1}(z) < 0\}$ and the sequence (λ_k) converges uniformly on compact subsets of $\mathbb{C}^2$ to $\hat{\lambda} : z \mapsto 2Rez^2 + 2ReK(z) + H(z^1, 0)$. Since J is diagonal, the sequence of structures $J_k := (d_k)_*(J)$ converges to J_{st} in the $\mathcal{C}^2$ norm on compact subsets of $\mathbb{C}^2$.

The discs h_k are J_k-holomorphic and the sequence $(h_k(0))$ is contained in $Q(0,1)$; passing to a subsequence we may assume that this converges to a point $p \in \overline{Q(0,1)}$. On the other hand, the function $\hat{\lambda} + A\hat{\lambda}^2$ is strictly J_{st}-plurisubharmonic on $Q(0,5)$ for a suitable constant $A > 0$. Since the structures J_k tend to J_{st}, the functions $\lambda_k + A\lambda_k^2$ are strictly J_k-plurisubharmonic on $Q(0,4)$ for every k large enough and their Levi forms admit a uniform lower bound with respect to k. By Proposition 4.4 the Kobayashi-Royden infinitesimal pseudometric on G_k admits the following lower bound : $K_{G_k}(z,v) \geq C|v|$ for any $z \in G_k \cap Q(0,3)$, $v \in \mathbb{C}^2$, with a positive constant C independent of k. Therefore, there exists a constant $C' > 0$ such that $|||(dh_k)_\zeta||| \leq C'$ for any $\zeta \in (1/2)\Delta$ satisfying $h_k(\zeta) \in G_k \cap Q(0,3)$. On the other hand, the sequence $(|h_k(\zeta_k)|)$ tends to $+\infty$. Denote by $[0,\zeta_k]$ the segment (in $\mathbb{C}$) joining the origin and ζ_k and let $\zeta'_k \in [0,\zeta_k]$ be the point the closest to the origin such that $h_k([0,\zeta'_k]) \subset G_k \cap \overline{Q(0,2)}$ and $h_k(\zeta'_k) \in \partial Q(0,2)$. Since $h_k(0) \in Q(0,1)$, we have $|h_k(0) - h_k(\zeta'_k)| \geq C''$ for some constant $C'' > 0$. Let $\zeta'_k = r_k e^{i\theta_k}$, $r_k \in]0,1[$. Then

$$|h_k(0) - h_k(\zeta'_k)| \leq \int_0^{r_k} |||(dh_k)_{te^{i\theta_k}}|||dt \leq C'r_k \to 0.$$

This contradiction proves Lemma 7.3. □

The statement of Lemma 7.3 remains true if we replace the unit disc Δ by the unit ball $\mathbb{B}_2$ in $\mathbb{C}^2$ equipped with an almost complex structure $\tilde{J}$ close enough (in the $\mathcal{C}^2$ norm) to J_{st}. For the proof it is sufficient to foliate $\mathbb{B}_2$ by $\tilde{J}$-holomorphic curves through the origin (in view of a smooth dependence on small perturbations of J_{st} such a foliation is a small perturbation of the foliation by complex lines through the origin, see [22]) and apply Lemma 7.3 to the foliation.

As a corollary we have the following

Lemma 7.4. *Let $(M,\tilde{J})$ be an almost complex manifold and let $F^k : M \to G$ be a sequence of $(\tilde{J},J)$-holomorphic maps. Assume that for some point $p^0 \in M$ we have $F^k(p) = (0,-\delta_k)$, $\delta_k \to 0$, and that the sequence (F^k) converges to 0 uniformly on compact subsets of M. Consider the rescaled maps $d_k \circ F^k$. Then for any compact subset $K \subset M$ the sequence of norms $(\|d_k \circ F^k\|_{\mathcal{C}^0(K)})$ is bounded.*

Proof of Lemma 7.4. It is sufficient to consider a covering of a compact subset of M by sufficiently small balls, similarly to [5], p.84. Indeed, consider a covering of K by the balls $p^j + r\mathbb{B}$, $j = 0,...,N$ where r is given by Lemma 7.3 and $p^{j+1} \in p^j + r\mathbb{B}$ for any j. For k large enough, we obtain that $F^k(p^0 + r\mathbb{B}) \subset Q(0, 2C\delta_k)$, and $F^k(p^1 + r\mathbb{B}) \subset Q(0, 4C^2\delta_k)$. Continuing this process we obtain that $F^k(p^N + r\mathbb{B}) \subset Q(0, 2^N C^N \delta_k)$. This proves Lemma 7.4. □

Now we return to the proof of Proposition 7.2. Lemma 7.4 implies that the sequence $(\hat{f}^k)$ is bounded (in the $\mathcal{C}^0$ norm) on any compact subset K of Σ. Covering K by small bidiscs, consider two transversal foliations by J-holomorphic curves on every bidisc. Since the restriction of $\hat{f}^k$ on every such curve is uniformly bounded in the $\mathcal{C}^0$-norm, it follows by the elliptic estimates that this is bounded in $\mathcal{C}^l$ norm for every l (see [28]). Since the bounds are uniform with respect to curves, the sequence $(\hat{f}^k)$ is bounded in every $\mathcal{C}^l$-norm. So the family $(\hat{f}^k)$ is relatively compact.

Step 2: Holomorphy of the limit maps. Let $(\hat{f}^{k_s})$ be a subsequence converging to a smooth map $\hat{f}$. Since f^{k_s} satisfies the holomorphy condition $\hat{J}'_{k_s} \circ d\hat{f}^{k_s} = d\hat{f}^{k_s} \circ J_{k_s}$, since $\hat{J}_{k_s}$ and $\hat{J}'_{k_s}$ converge to J_{st}, we obtain, passing to the limit in the holomorphy condition, that $\hat{f}$ is holomorphic with respect to J_{st}.

Step 3: Biholomorphy of $\hat{f}$. Since $\hat{f}(0,-1) = (0,-1) \in \Sigma'$ and Σ' is defined by a plurisubharmonic function, it follows by the maximum principle that $\hat{f}(\Sigma) \subset \Sigma'$ (and not just a subset of $\bar{\Sigma}'$). Applying a similar argument to the sequence $(\hat{f}^k)^{-1}$ of inverse map, we obtain that this converges (after extraction of a subsequence) to the inverse of $\hat{f}$.

Finally the domain Σ (resp. Σ') is biholomorphic to $\mathbb{H}$ by means of the transformation $(z^1, z^2) \mapsto (z^1, z^2 + K(z^1, 0))$ (resp. $(z^1, z^2) \mapsto (z^1, z^2 + K'(z^1, 0))$). Since a biholomorphism of $\mathbb{H}$ fixing the point $(0,-1)$ has the form $(e^{i\theta} z^1, z^2)$ (see, for instance, [9]), $\hat{f}$ is conjugated to this transformation by the above quadratic biholomorphisms of $\mathbb{C}^2$. Hence :

$$\frac{\partial \hat{f}^2}{\partial z^2}(0,-1) = 1. \tag{7.1}$$

This property will be used in the next Section. □

8. Boundary behavior of the tangent map

We suppose that we are in the local situation described at the beginning of the previous section. Here we prove two statements concerning the boundary behavior of the tangent map of f near Γ. They are obvious if f is of class $\mathcal{C}^1$ up to Γ. In the general situation, their proofs require the scaling method of the previous section. Let $p \in \Gamma$. After a local change of coordinates z we may assume that $p = 0$, $J(0) = J_{st}$ and J is assumed to be diagonal. In the z coordinates, we consider a base X of (1,0) (with respect to J) vector fields defined in Subsection 3.3. Recall that $X_2 = \partial/\partial z^2 + a(z)\partial/\bar{\partial} z^2$, $a(0) = 0$, $X_1(0) = \partial/\partial z^1$ and at every point z^0, $X_1(z^0)$ generates the holomorphic tangent space $H^J_z(\partial D - t)$, $t \geq 0$. If we return to the initial coordinates and move the point $p \in \Gamma$, we obtain for every p a basis X_p of $(1,0)$ vector fields, defined in a neighborhood of p. Similarly, we define the basis X'_q for $q \in \partial D'$.

The elements of the matrix of the tangent map df_z in the bases $X_p(z)$ and $X'_{f(p)}(z)$ are denoted by $A_{js}(p,z)$. According to Proposition 3.5 the function $A_{22}(p,\cdot)$ is upper bounded on D.

Proposition 8.1. *We have:*

(a) *Every cluster point of the function $z \mapsto A_{22}(p,z)$ (in the notation of Proposition 3.5) is real when z tends to a point $p \in \partial D$.*

(b) *For $z \in D$, let $p \in \Gamma$ such that $|z - p| = dist(z, \Gamma)$. There exists a constant A, independent of $z \in D$, such that $|A_{22}(p,z)| \geq A$.*

The proof of these statements use the above scaling construction. So we use the notations of the previous section.

Proof of Proposition 8.1. (a) Suppose that there exists a sequence of points (p^k) converging to a boundary point p such that $A_{22}(p,\cdot)$ tends to a complex number a. Applying the above scaling construction, we obtain a sequence of maps $(\hat{f}^k)_k$. Consider the two basis $\hat{X}^k := \delta_k^{1/2}((\phi_k^{-1}) \circ T^k)(X_1), \delta_k((\phi_k^{-1}) \circ T^k)(X_2))$ and $(\hat{X}')^k := (\varepsilon_k^{-1/2}((\psi_k^{-1}) \circ T'^k)(X'_1), \varepsilon_k^{-1}((\psi_k^{-1}) \circ T'^k)(X'_2))$. These vector fields tend to the standard (1,0) vector field base of $\mathbb{C}^2$ as k tends to ∞. Denote by $\hat{A}^k_{js}$ the elements of the matrix of $d\hat{f}^k(0,-1)$. Then $A^k_{22} \to (\partial \hat{f}^2/\partial z^2)(0,-1) = 1$, according to (7.1).

On the other hand, $A^k_{22} = \varepsilon_k^{-1}\delta_k A_{22}$ and tends to a by the boundary distance preserving property (Proposition 3.2). This gives the statement.

(b) Suppose that there is a sequence of points (p^k) converging to the boundary such that A_{22} tends to 0. Repeating precisely the argument of (a), we obtain that $(\partial \hat{f}^2/\partial z^2)(0,-1) = 0$; this contradicts (7.1). $\square$

In order to establish the next proposition, it is convenient to associate a wedge with the totally real part of the conormal bundle $\Sigma_J(\partial D)$ of ∂D as edge. Consider in $\mathbb{R}^4 \times \mathbb{R}^4$ the set $S = \{(z,L) : dist((z,L),\Sigma_J(\partial D)) \leq dist(z,\partial D), z \in D\}$. Then, in a neighborhood U of any totally real point of $\Sigma_J(\partial D)$, the set S contains a wedge W_U with $\Sigma_J(\partial D) \cap U$ as totally real edge.

Proposition 8.2. *Let K be a compact subset of the totally real part of the conormal bundle $\Sigma_J(\partial D)$. Then the cluster set of the cotangent lift $\tilde{f}$ of f on the conormal bundle $\Sigma(\partial D)$, when (z,L) tends to $\Sigma_J(\partial D)$ along the wedge W_U, is relatively compactly contained in the totally real part of $\Sigma(\partial D')$.*

Proof of Proposition 8.2. Let (z^k, L^k) be a sequence in W_U converging to $(0, \partial_J \rho(0)) = (0, dz^2)$. Set $g = f^{-1}$. We shall prove that the sequence of linear forms $Q^k := {}^t dg(w^k)L^k$, where $w^k = f(z^k)$, converges to a linear form which up to a *real* factor (in view of Part (a) of Proposition 8.1) coincides with $\partial_J \rho'(0) = dz^2$ (we recall that ^{t} denotes the transposed map). It is sufficient to prove that the first component of Q^k with respect to the dual basis (ω_1, ω_2) of X tends to 0 and the second one is bounded below from the origin as k tend to infinity. The map X being of class $\mathcal{C}^1$ we can replace $X(0)$ by $X(w^k)$. Since $(z^k, L^k) \in W_U$, we have $L^k = \omega_2(z^k) + O(\delta_k)$, where δ_k is the distance from z^k to the boundary. Since $|||dg_{w^k}||| = 0(\delta_k^{-1/2})$, we have $Q^k = {}^t dg_{w^k}(\omega_2(z^k)) + O(\delta_k^{1/2})$. By Proposition 3.5, the components of ${}^t dg_{w^k}(\omega_2(z^k))$ with respect to the basis $(\omega_1(z^k), \omega_2(z^k))$ are the elements of the second line of the matrix dg_{w^k} with respect to the basis $X'(w^k)$ and $X(z^k)$. So its first component is $0(\delta_k^{1/2})$ and tends to 0 as k tends to infinity. Finally the component A^k_{22} is bounded below from the origin by Part (b) of Proposition 8.1. $\square$

Proof of Theorem 1.1. In view of Proposition 8.2, we may apply Proposition 5.1 to the cotangent lift $\tilde{f}$ of f. This gives the statement of Theorem 1.1. $\square$

9. Appendices

9.1. **Appendix 1 : Uniform estimates of the Kobayashi-Royden metric.** We prove Proposition 4.4, restated in Proposition 9.2. Our method is based on Sibony's approach [28].

For our construction we need plurisubharmonic functions with logarithmic singularities on almost complex manifolds. The function $\log|z|$ is not J-plurisubharmonic in a neighborhood of the origin even if the structure J is $\mathcal{C}^2$ close to the standard one. However, for a suitable positive constant $A > 0$ the function $\log|z| + A|z|$ is J-plurisubharmonic on the unit ball $\mathbb{B}$ for any almost complex structure J with $\|J - J_{st}\|_{C^2(\bar{\mathbb{B}})}$ small enough. This useful observation due to E.Chirka can be easily established by direct computation of the Levi form, see [15] (we point out that the Levi form of $|z|$ goes to $+\infty$ at the origin neutralizing the growth of the logarithm). This implies the following :

Lemma 9.1. *Let $r < 1$ and let θ_r be a smooth nondecreasing function on $\mathbb{R}^+$ such that $\theta_r(s) = s$ for $s \leq r/3$ and $\theta_r(s) = 1$ for $s \geq 2r/3$. Let (M,J) be an almost complex manifold, and let p be a point of M. Then there exists a neighborhood U of p, positive constants $A = A(r)$, $B = B(r)$ and a diffeomorphism $z : U \to B$ such that $z(p) = 0$, $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$ and the function $\log(\theta_r(|z|^2)) + \theta_r(A|z|) + B|z|^2$ is J-plurisubharmonic on U.*

The main estimate of the Kobayashi-Royden metric is given by the following

Proposition 9.2. *Let D be a domain in an almost complex manifold (M, J), let $p \in \bar{D}$, let U be a neighborhood of p in M (not necessarily contained in D) and let $z : U \to \mathbb{B}$ be the diffeomorphism given by Lemma 9.1. Let u be a $\mathcal{C}^2$ function on $\bar{D}$, negative and J-plurisubharmonic on D. We assume that $-L \leq u < 0$ on $D \cap U$ and that $u - c|z|^2$ is J-plurisubharmonic on $D \cap U$, where c and L are positive constants. Then there exists a neighborhood U' of p and a constant $c' > 0$, depending on c and L only, such that :*

$$K_{(D,J)}(q, v) \geq c' \frac{\|v\|}{|u(q)|^{1/2}}, \tag{9.1}$$

for every $q \in D \cap U'$ and every $v \in T_q M$.

Proof of proposition 9.2. Step 1: Local hyperbolicity. We prove the following rough estimate

$$K_{(D,J)}(q, v) \geq s K_{(D \cap U, J)}(q, v) \tag{9.2}$$

which allows to localize the proof (s is a positive constant). Let $0 < r < 1$ be such that the set $V_1 := \{q \in U : |z(q)| \leq \sqrt{r}\}$ is relatively compact in U and let θ_r be a smooth nondecreasing function on $\mathbb{R}^+$ such that $\theta_r(s) = s$ for $s \leq r/3$ and $\theta_r(s) = 1$ for $s \geq 2r/3$. According to Lemma 9.1, there exist uniform positive constants A and B such that the function $\log(\theta_r(|z - z(q)|^2)) + \theta_r(A|z - z(q)|) + B|z|^2$ is J-plurisubharmonic on U for every $q \in V$. By assumption the function $u - c|z|^2$ is J-plurisubharmonic on $D \cap U$. Set $\tau = 2B/c$ and define, for every point $q \in V$, the function Ψ_q by :

$$\begin{cases} \Psi_q(z) &= \theta_r(|z - z(q)|^2) \exp(\theta_r(A|z - z(q)|)) \exp(\tau u(z)) \text{ if } z \in D \cap U, \\ \\ \Psi_q &= \exp(1 + \tau u) \text{ on } D \backslash U. \end{cases}$$

Then for every $0 < \varepsilon \leq B$, the function $\log(\Psi_q) - \varepsilon |z|^2$ is J-plurisubharmonic on $D \cap U$ and hence Ψ_q is J-plurisubharmonic on $D \cap U$. Since Ψ_q coincides with $\exp(\tau u)$ outside U, it is globally J-plurisubharmonic on D.

Let $f \in \mathcal{O}_J(\Delta, D)$ be such that $f(0) = q \in V_1$ and $(\partial f/\partial x)(0) = v/\alpha$ where $v \in T_q M$ and $\alpha > 0$. For ζ sufficiently close to 0 we have $f(\zeta) = q + df_0(\zeta) + \mathcal{O}(|\zeta|^2)$. Setting $\zeta = \zeta_1 + i\zeta_2$ and using the J-holomorphy condition $df_0 \circ J_{st} = J \circ df_0$, we may write $df_0(\zeta) = \zeta_1 df_0(\partial/\partial x) + \zeta_2 J(df_0(\partial/\partial x))$. Consider the function $\varphi(\zeta) = \Psi_q(f(\zeta))/|\zeta|^2$ which is subharmonic on $\Delta \backslash \{0\}$. Since $\varphi(\zeta) = |f(\zeta) - q|^2/|\zeta|^2 \exp(A|f(\zeta) - q|) \exp(\tau u(f(\zeta)))$ for ζ close to 0 and $\|df_0(\zeta)\| \leq |\zeta|(\|I + J\| \, \|df_0(\partial/\partial x)\|)$, we obtain that $\limsup_{\zeta \to 0} \varphi(\zeta)$ is finite. Moreover setting $\zeta_2 = 0$ we have $\limsup_{\zeta \to 0} \varphi(\zeta) \geq \|df_0(\partial/\partial x)\|^2 \exp(-2B|u(q)|/c)$. Applying the maximum principle to a subharmonic extension of φ on Δ we obtain the inequality $\|df_0(\partial/\partial x)\|^2 \leq \exp(1 + 2B|u(q)|/c)$. Hence, by definition of the Kobayashi-Royden infinitesimal pseudometric, we obtain for every $q \in D \cap V_1$, $v \in T_q M$:

$$K_{(D,J)}(q, v) \geq \left(\exp\left(-1 - 2B\frac{|u(q)|}{c}\right)\right)^{1/2} \|v\|.$$

We denote by $d^K_{(M,J)}$ the integrated pseudodistance of the Kobayashi-Royden infinitesimal pseudometric. According to the almost complex version of Royden's theorem [21], it coincides with the usual Kobayashi pseudodistance on (M, J) defined by means of J-holomorphic discs. Consider now the Kobayashi ball $B_{(D,J)}(q, \alpha) = \{w \in D : d^K_{(D,J)}(w, q) < \alpha\}$. It follows from Lemma 2.2 of [6] (whose proof is identical in the almost complex setting) that there is a neighborhood V of p, relatively compact in V_1 and a positive constant $s < 1$, independent of q, such that for every $f \in \mathcal{O}_J(\Delta, D)$ satisfying $f(0) \in D \cap V$ we have $f(s\Delta) \subset D \cap U$. This gives the inequality (9.2).

Step 2. It follows from (9.2) that there is a neighborhood V of p in $\mathbb{C}^n$, contained in U and a positive constant s such that $D_{(D,J)}(q,v) \geq sK_{(D\cap U,J)}(q,v)$ for every $q \in V,\ v \in T_qM$. Consider a positive constant r that will be specified later and let θ be a smooth nondecreasing function on $\mathbb{R}^+$ such that $\theta(x) = x$ for $x \leq 1/3$ and $\theta(x) = 1$ for $x \geq 2/3$. Restricting U if necesssary it follows from Lemma 9.1 that the function $\log(\theta(|(z-q)/r|^2)) + A|z-q| + B|(z-q)/r|^2$ is J-plurisubharmonic on $D \cap U$, independently of q and r.

Consider now the function $\Psi_q(z) = \theta\left(\frac{|z-q|^2}{r^2}\right)\exp(A|z-q|)\exp(\tau u(z))$ where $\tau = 1/|u(q)|$ and $r = (2B|u(q)|/c)^{1/2}$. Since the function $\tau u - 2B|(z-q)/r|^2$ is J_{st}-plurisubharmonic, we may assume, shrinking U if necessary, that the function $\tau u - B|(z-q)/r|^2$ is J-plurisubharmonic on $D\cap U$. Hence the function $\log(\Psi_q)$ is J-plurisubharmonic on $D \cap U$. Let $q \in V$, let $v \in T_qM$ and let $f : \Delta \rightarrow D$ be a J-holomorphic map be such that $f(0) = q$ and $df_0(\partial/\partial x) = v/\alpha$ where $\alpha > 0$. We have $f(\zeta) = q + df_0(\zeta) + \mathcal{O}(|\zeta|^2)$. Setting $\zeta = \zeta_1 + i\zeta_2$ and using the J-holomorphy condition $df_0 \circ J_{st} = J \circ df_0$, we may write $df_0(\zeta) = q + \zeta_1 df_0(\partial/\partial x) + \zeta_2 J(df_0(\partial/\partial x))$. Consider the function $\varphi(\zeta) = \Psi_q(f(\zeta))/|\zeta|^2$ which is subharmonic on $\Delta\backslash\{0\}$. Since $\varphi(\zeta) = |f(\zeta)-q|^2/(r^2|\zeta|^2)\exp(\tau u(f(\zeta)))$ and $|df_0(\zeta)| \leq |\zeta|(\|I + J\|\,\|df_0(\partial/\partial x)\|$, we obtain that $\limsup_{\zeta\rightarrow 0}\varphi(\zeta)$ is finite. Setting $\zeta_2 = 0$ we obtain $\limsup_{\zeta\rightarrow 0}\phi(\zeta) \geq \|v\|^2\exp(2)/(r^2\alpha^2)$. There exists a positive constant C', independent of q, such that $|z-q| \leq C'$ on D. Applying the maximum principle to a subharmonic extension of ϕ on Δ, we obtain the inequality

$$\alpha \geq \sqrt{\frac{c}{2B\exp(1+AC')}}\|v\|^2/|u(q)|^{1/2}.$$

This completes the proof. □

9.2. **Appendix 2 : Canonical lift of an almost complex structure to the cotangent bundle.** We recall the definition of the canonical lift of an almost complex structure J on M to the cotangent bundle T^*M, following [32]. Set $m = 2n$. We use the following notations. Suffixes A,B,C,D take the values 1 to $2m$, suffixes $a, b, c, \ldots, h, i, j, \ldots$ take the values 1 to m and $\bar{j} = j + m$, ... The summation notation for repeated indices is used. If the notation (ε_{AB}), (ε^{AB}), (F_B^A) is used for matrices, the suffix on the left indicates the column and the suffix on the right indicates the row. We denote local coordinates on M by $(x^1, \ldots, x^n)$ and by $(p_1, \ldots, p_n)$ the fiber coordinates.

Recall that the cotangent space $T^*(M)$ of M possesses the *canonical contact form* θ given in local coordinates by $\theta = p_i dx^i$. The cotangent lift φ^* of any diffeomorphism φ of M is contact with respect to θ, that is θ does not depend on the choice of local coordinates on $T^*(M)$.

The exterior derivative $d\theta$ of θ defines *the canonical symplectic structure* of $T^*(M)$: $d\theta = dp_i \wedge dx^i$ which is also independent of local coordinates in view of the invariance of the exterior derivative. Setting $d\theta = (1/2)\varepsilon_{CB}dx^C \wedge dx^B$ (where $dx^{\bar{j}} = dp_j$), we have

$$(\varepsilon_{CB}) = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Denote by (ε^{BA}) the inverse matrix and write ε^{-1} for the tensor field of type (2,0) whose component are (ε^{BA}). By construction, this definition does not depend on the choice of local coordinates.

Let now E be a tensor field of type (1,1) on M. If E has components E_i^h and E_i^{*h} relative to local coordinates x and x^* repectively, then $p_a^* E_i^{*\,a} = p_a E_j^b \frac{\partial x^j}{\partial x^{*i}}$. If we interpret a change of coordinates as a diffeomorphism $x^* = x^*(x) = \varphi(x)$ we denote by E^* the direct image of the tensor E under the action of φ. In the case where E is an almost complex structure (that is $E^2 = -Id$), then φ is a biholomorphism between (M, E) and (M, E^*). Any (1,1) tensor field E on M canonically defines a contact form on E^*M via $\sigma = p_a E_b^a dx^b$. Since $(\varphi^*)^*(p_a^* E_b^{*\,a} dx^{*b}) = \sigma$, σ does not depend on a choice of local coordinates (here φ^* is the cotangent lift of φ). Then this canonically defines the symplectic form

$$d\sigma = p_a \frac{\partial E_b^a}{\partial x^c} dx^c \wedge dx^b + E_b^a dp_a \wedge dx^b.$$

The cotangent lift φ^* of a diffeomorphism φ is a symplectomorphism for $d\sigma$. We may write $d\sigma = (1/2)\tau_{CB} dx^C \wedge dx^B$ where $x^{\bar{i}} = p_i$; so we have

$$\tau_{ji} = p_a \left(\frac{\partial E_i^a}{\partial x^j} - \frac{\partial E_j^a}{\partial x^i} \right), \tau_{\bar{j}i} = E_i^j, \tau_{j\bar{i}} = -E_j^i, \tau_{\bar{j}\bar{i}} = 0.$$

We write $\widehat{E}$ for the tensor field of type (1,1) on $T^*(M)$ whose components $\widehat{E}_B^A$ are given by $\widehat{E}_B^A = \tau_{BC}\varepsilon^{CA}$. Thus $\widehat{E}_i^h = E_i^h$, $\widehat{E}_{\bar{i}}^h = 0$ and $\widehat{E}_i^{\bar{h}} = p_a \left(\frac{\partial E_i^a}{\partial x^j} - \frac{\partial E_j^a}{\partial x^i} \right)$, $\widehat{E}_{\bar{i}}^{\bar{h}} = E_h^i$. In the matrix form we have

$$\widehat{E} = \begin{pmatrix} E_i^h & 0 \\ p_a \left(\frac{\partial E_i^a}{\partial x^j} - \frac{\partial E_j^a}{\partial x^i} \right) & E_h^i \end{pmatrix}.$$

By construction, the complete lift $\widehat{E}$ has the following *invariance property* : if φ is a local diffeomorphism of M transforming E to E', then the direct image of $\widehat{E}$ under the cotangent lift $\psi := \varphi^*$ is $\widehat{E'}$. In general, $\widehat{E}$ is not an almost complex structure, even if E is. Moreover, one can show [32] that $\widehat{J}$ is a complex structure if and only if J is integrable. One may however construct an almost complex structure on $T^*(M)$ as follows.

Let S be a tensor field of type (1,s) on M. We may consider the tensor field γS of type $(1, s-1)$ on T^*M, defined in local canonical coordinates on T^*M by the expression

$$\gamma S = p_a S_{i_s \ldots i_2 i_1}^{\ \ a} dx^{i_s} \otimes \cdots \otimes dx^{i_2} \otimes \frac{\partial}{\partial p_{i_1}}.$$

In particular, if T is a tensor field of type (1,2) on M, then γT has components

$$\gamma T = \begin{pmatrix} 0 & 0 \\ p_a T_{ji}^{\ a} & 0 \end{pmatrix}$$

in the local canonical coordinates on T^*M.

Let F be a (1,1) tensor field on M. Its Nijenhuis tensor N is the tensor field of type (1,2) on M acting on two vector fields X and Y by

$$N(X,Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y].$$

By NF we denote the tensor field acting by $(NF)(X,Y) = N(X, FY)$. The following proposition is proved in [32] (p.256).

Proposition 9.3. *Let J be an almost complex structure on M. Then*

$$\tilde{J} := \widehat{J} + (1/2)\gamma(NF) \tag{9.3}$$

is an almost complex structure on the cotangent bundle $T^*(M)$.

We stress that the definition of the tensor $\tilde{J}$ is independent of the choice of coordinates on T^*M. Therefore if ϕ is a biholomorphism between two almost complex manifolds (M, J) and (M', J'), then its cotangent lift is a biholomorphism between $(T^*(M), \tilde{J})$ and $(T^*(M'), \tilde{J}')$. Indeed one can view ϕ as a change of coordinates on M, J' representing J in the new coordinates. The cotangent lift ϕ^* defines a change of coordinates on T^*M and $\tilde{J}'$ represents $\tilde{J}$ in the new coordinates. So the assertion (i) of Proposition 6.2 holds. Property (ii) of Proposition 6.2 is immediate in view of the definition of $\tilde{J}$ given by (9.3).

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Bernard Coupet
C.M.I.
39, rue Joliot-Curie,
13453 Marseille Cedex 13
FRANCE

coupet@cmi.univ-mrs.fr

Hervé Gaussier
C.M.I.
39, rue Joliot-Curie,
13453 Marseille Cedex 13
FRANCE

gaussier@cmi.univ-mrs.fr

Alexandre Sukhov
U.S.T.L.
Cité Scientifique
59655 Villeneuve d'Ascq Cedex
FRANCE

sukhov@agat.univ-lille1.fr